Algebra · Book Two

REVISED EDITION

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In this revision of the Welchons-Krickenberger Algebra, Book Two, changes have been made in the development of subject matter, sequence of topics, and gradation of exercises and problems wherever they seemed to make the book easier for the teacher and student to use The treatment of linear functions has been extended and a brief chapter on statistics has been added

This book is intended for students who have completed the study of first-year algebra It can be used either before or after the study of plane geometry

In preparing the text the authors have aimed to arouse the student's interest in mathematics and to maintain that interest throughout the course They have tried to present the subject matter so that the student must think for himself

Some of the special features of the book are the following

Clear Presentation

The explanations and discussions are so written that they may be readily understood by the students Numerous illustrations and examples assist in the understanding of new concepts, principles, and procedures Pitfalls are anticipated and care has been used to avoid them New terms are carefully defined and new operations thoroughly explained

Style

The book is written to the student Its language is simple and easily understood Sentences and paragraphs are short

Visual Aids

The graph is used as a visual aid in the study of formulas and other equations It is used as a means rather than as an objective Other diagrams likewise are used where they will serve a real purpose Color has been used as a teaching aid and to add to the attractiveness of the book

Motivation

Various means have been used to create and maintain interest New concepts are presented as extensions of familiar ones The need for mathematics is shown by articles, discussions, problems, and pictures

Functional Relations

More than the usual treatment of functional relations is presented Throughout the book students are led to discover the dependence of one quantity upon another

Applications

The text contains applications of algebra to geometry, commerce, science, and industry The applied problems do not require extensive knowledge in any of these fields Special attention is given to the preparation of the pupils for physics and chemistry

Reviews and Tests

The book contains adequate reviews and tests It includes chapter and cumulative reviews, chapter check lists, and chapter and cumulative tests The tests can be used as reviews, self-tests, or as sample tests

Individual Differences

The book provides for individual differences by including topics and exercises labeled A, B, and C The A exercises and topics are intended for all students The B exercises and topics are for those students who wish to do more than the minimum requirement Those marked by a C are for pupils especially gifted in mathematics

Flexibility

The text was designed for a full year's course but so arranged that it can be used effectively for a terminal third-semester course

Most schools will be able to cover Chapters 1–9 in one semester and Chapters 10–17 in a second semester Because of differences in classes the authors have not attempted to specify the time needed for any one topic The following selections are only suggestive

A Minimum Course

First semester Chapters 1–7, omitting topics and exercises labeled B and C

Second semester Chapters 8–14, omitting topics and exercises labeled B and C

A Medium Course

First semester Chapters 1–9
Second semester Chapters 10–17

A Maximum Course

First semester Chapters 1–11
Second semester Chapters 12–20, and if time permits, Chapters 21 and 22

Meeting Requirements

Every effort has been made to have the text meet the requirements of present-day needs It is in accord with the Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, the recommendations of the College Entrance Examination Board, and various state syllabi

The Authors

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Algebra · Book Two

CHAPTER 1

Fundamental Operations

In this chapter you look at numbers and rules for operating with them

In your first course in algebra you learned many facts about num bers and their relationships In this second course you will study new numbers and new relationships and will deepen your understanding of the facts you learned in your first course You will be much more able to understand this second course in algebra if you give some thought to how numbers came into being and why

The Beginnings of Number[A]

Man s first numbers were the whole numbers 1 2 3 At that time no one had yet dreamed of fractions or negative numbers, or any of the other numbers you will study in this second course in alge- bra Had you appeared at a gathering of the most brilliant students of that day with the following simple test, every student would have failed to do at least six of the exercises Can you determine which exercises all the students would have found impossible?

Perform the indicated operations

TEST

1 $14 + 9$	6 $\frac{3}{4}$ of 8	11 $163 + 27 - 46$
2 $8 \div 2$	7 $11 - 4$	12 $3 - 5 + 6$
3 $7 \div 3$	8 $6 - 9$	13 $4 - 2 - 3$
4 $4\frac{1}{2} + 3\frac{1}{2}$	9 $2 \div 3$	14 12×4
5 6×9	10 $3 \div 2$	15 $6\ 2 + 9\ 3 + 4\ 6$

We now call the first numbers the <u>natural numbers</u> Why do you think they are so called?

As man had worked with the natural numbers four patterns of operating with them had come to be considered more fundamental than others We now call these operations addition subtraction, multiplication, and division You know that addition and subtrac tion are inverse operations that is, operations which can be used to nullify each other For example, if we subtract 5 from your algebra grade and then add 5, your final grade is the same as the original You know that multiplication and division are also inverse operations For example, $(3 \times 2) \div 2 = 3$ We must remember, however, that a product with a zero factor has no inverse because division by zero is meaningless

As time passed people noticed that the number system possessed certain relationships They saw that

Addition and multiplication are each <u>commutative</u>

That is, numbers may be added in any order or multiplied in any order Thus,

$$3+5+7=3+7+5=5+3+7=15$$
$$7\times 4\times 3=7\times 3\times 4=4\times 3\times 7=84$$

Addition and multiplication are each associative

That is, the sum of three or more numbers is the same in whatever manner the numbers are grouped, and the product of three or more numbers is the same in whatever manner the numbers are grouped Thus,

$$3+5+7+8=(3+5)+(7+8)=8+15=23$$
$$3+5+7+8=(7+5)+(3+8)=12+11=23$$
$$4\times 5\times 3=(4\times 5)\times 3=20\times 3=60$$
$$4\times 5\times 3=4\times(5\times 3)=4\times 15=60$$

Multiplication is distributive with respect to addition

That is, when the sum of two or more numbers is to be multiplied by a number, each of them may be multiplied by it separately and the products added Thus,

$$4(7+2)=28+8=36$$

These facts were so much a part of the number system which man had invented that he had no control over them He called them laws, meaning, in a sense, laws of the nature of the number system We can prove these laws true for the system of natural numbers

EXERCISES

(A)

In Exercises 1–3, find the missing word which will make the statement true

1. The statement $5+3=3+5$ illustrates the fact that the __?__ law holds for addition

2. The statement "$5-3$ does not equal $3-5$" illustrates the fact that the __?__ law does not hold for subtraction

3. The statement "$6\div 2$ does not equal $2\div 6$" illustrates the fact that the __?__ law does not hold for division

4 The formula for the volume of a rectangular solid is $V=lwh$ Show how the associative law makes it possible to find the volume in three ways

5. The formula for the perimeter of a rectangle may be written as $p = 2(l + w)$ If $l = 10$ inches and $w = 6$ inches, find the value of p by use of the distributive law Find the value of p without using the distributive law

Man Invents Fractions [A]

As long as the number system consisted of only the natural numbers some divisions were impossible It was possible to perform such divisions as $8 \div 2$ because the result was another one of the natural numbers It was even possible, in a sense, to perform such uneven divisions as $7 \div 2$, for both the quotient and the remainder were natural numbers It was, however, impossible to perform such divisions as $1 \div 2$ for there was no number to express the result Had there been books in that day, the books would have said Division is impossible when the divisor is larger than the dividend

But all around him man saw things broken into two parts or more parts There were times when it was convenient for him to make three equal portions from what had originally been two Gradually it dawned upon him that he needed a new number to express such divisions—that division of a small number by a large one is not impossible, it is merely excluded when we limit our number system to the natural numbers

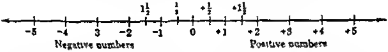
A new number called for a new symbol, and since all the previously known numbers were expressed by the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, it seemed right to express the new number with the same symbols The issue was decided by writing $1 \div 2 = \frac{1}{2}$ The new numbers were called *fractions* The next task was to make rules for adding, subtracting, multiplying, and dividing fractions—rules which would fit them into the already existing number system and make them obey all its laws without exception You know the rules and have used them again and again

Man Invents Negative Numbers [A]

For thousands of years men felt it absurd to try to subtract a large number from a small one, but there came a time when someone began to see the possibilities for a new number that would make such subtractions possible We now call these numbers negative numbers and symbolize them by the ordinary number symbols preceded by minus signs, thus $-1, -2, -3, \cdot$

The new numbers not only made it possible to subtract a large number from a small, they made it possible for the number system to express direction. Direction is very much a part of our lives. We gain money and we lose it, we walk a certain distance in one direction, and then we retrace our steps in the opposite direction, the temperature rises and falls, and all this suggests the importance of a number system which not only tells *how much*, but also *in what direction*.

If we designate the natural numbers and the fractions which are greater than zero as positive numbers, in contrast to the newer negative numbers, we can arrange the system along a line, thus



On our scale we can see the positive numbers extending in one direction from zero and the negative numbers extending in the opposite direction. Zero is neither positive nor negative.

If we consider only the size of a number and not its direction, that is, if we ignore its sign, we are considering its absolute value. The numbers -3 and $+3$ have the same absolute value. The number -7 has a greater absolute value than the number $+2$.

Having invented negative numbers, man was forced to devise rules for adding, subtracting, multiplying, and dividing them—rules which would fit them into the already existing number system, and make them conform to all its laws and relationships. You studied those rules in your first course in algebra, but they are repeated here to enable you to review them.

Addition of Positive and Negative Numbers[A]

Rules for Adding Positive and Negative Numbers

1. To add two numbers having the same sign,
   use the common sign
   and find the sum of their absolute values.
2. To add two numbers having opposite signs,
   use the sign of the number having the greater absolute value
   and find the difference of their absolute values.

To help yourself remember these rules, think of positive numbers as gains and of negative numbers as losses, or remember how you add scores in certain games

Examples $(+4)+(-7)=-3, \quad (-5)+(-3)=-8$
$(-10)+(-10)=-20, \quad -7+7=0$

The numbers that are added are called <u>addends</u> In the addition $5+8=13$ the addends are 5 and 8 The result of adding is called the <u>sum</u>

(A) **EXERCISES**

Add (or combine)

	a	b	c	d	e	f	g
1	4	− 10	+ 8	− 11	4	5	− 7
	− 3	− 4	3	3	− 9	10	− 4
2	6	7	− 4	− 10	− 12	10	− 11
	− 5	− 3	− 5	4	+ 12	0	1
3	$\frac{1}{2}$	$\frac{2}{5}$	$-\frac{4}{7}$	$-\frac{2}{5}$	$\frac{3}{4}$	$-\frac{7}{8}$	$6\frac{1}{4}$
	$-\frac{3}{4}$	$-\frac{3}{5}$	$-\frac{3}{7}$	$+\frac{2}{5}$	$-\frac{1}{4}$	$-\frac{3}{8}$	-7
4	− 1 0	2 7	7 6	− 8 1	− 10 0	− 9 4	− 7
	− 0 3	− 1 5	2 4	4 1	− 2 1	− 2 6	3 4

Simplify

5 $(+7)+(-6)$

6 $(-3)+(-20)$

7. $-8-9$

8. $-4-60$

9 $(-2)+(-15)$

10 $\frac{1}{2}+(-\frac{1}{2})$

11. $5\ 6+(-10)$

12 $-7\ 3+(-2\ 7)$

Remember that the commutative and associative laws hold for negative numbers When you are to add several numbers, those laws make it possible for you to add the positive numbers first, to add the negative numbers next, and then to combine the two sums

Subtraction of Positive and Negative Numbers (A)

Rule for Subtracting Positive and Negative Numbers

To subtract one number from another,
change the sign of the subtrahend and proceed as in addition.

The subtrahend is the number that is subtracted and the minuend is the number from which the subtrahend is subtracted The difference or remainder is the result of the subtraction

Example Subtract -7 from -1

Solution Change -7 to $+7$ Then add $+7$ to -1 The result, which is the difference is $+6$

EXERCISES

[A]

Subtract

	a	b	c	d	e	f	g
1	6	5	4	3	-7	-4	6
	4	-3	-1	10	2	-7	8
2	4	$+7$	-6	-5	10	-9	0
	8	-7	-6	5	-40	8	6
3	1 7	-3 2	4 0	1 0	6 0	0	5 6
	0 4	-1 2	-0 8	-5 4	0 21	8 4	-8 4
4	$\frac{1}{2}$	-32	$4\frac{1}{2}$	$1\frac{3}{4}$	8	$2\frac{3}{4}$	$-7\frac{3}{8}$
	$\frac{1}{4}$	$-4\frac{1}{5}$	$-3\frac{1}{2}$	$3\frac{1}{2}$	6 4	$-4\frac{1}{3}$	$-1\frac{1}{4}$

Subtract as indicated

5 $7-(-4)$

6 $10-(+3)$

7 $(-8)-4$

8 $0-(-9)$

9 $7-(-6)$

10 $4-(-2)$

11 $4-(-4)$

12 $-30-(-20)$

13 $1-(-100)$

14 $-7\frac{1}{2}-(-2\frac{1}{2})$

15 $10-(-10)$

16 $5-(+12)$

Simplify

17 $8+7-(-6)$

18 $3-(-6)+(-4)$

19 $1-(-1)+1$

20 $4\frac{1}{2}-2\frac{1}{4}+(-6\frac{1}{4})$

21 $8-9-7$

22 $10-(-8)+3$

Multiplication of Positive and Negative Numbers [A]

In multiplication the number that is multiplied is the multiplicand, and the number that does the multiplying is the multiplier The result is the product

The signs denoting multiplication are the times sign ($\times$) and the dot ( ) These signs are usually omitted unless they are necessary. For example, $4x$ means "4 times x" Also, xy means $x \cdot y$, and $7(x + 3)$ means "7 times the sum of x and 3" The multiplication sign cannot be omitted between two numerals, as in the expression 7×8 Can you tell why?

Law of Signs for Multiplication

1. The product of two numbers with <u>like</u> signs is a positive number
2. The product of two numbers with <u>unlike</u> signs is a negative number

Examples $(-7)(-3) = +21$ $\quad -5 \times 3 = -15$
$4 \times (-6) = -24$ $\quad -4 \times (-7) = +28$

[A] **EXERCISES**

Multiply

	a	b	c	d	e	f	g
1	4 -3	5 -2	-6 4	8 2	-10 1	-5 -3	-2 4
2	5 -6	6 -7	-7 8	-9 -1	-1 1	-2 -2	-13 2

3 $(-4)(5)$ 11 $-12(3)$ 19 $2 \times 5 \times 6$
4. $(-2)(-3)$ 12 $-10(-10)$ 20 $4 \times 10 \times \frac{1}{2}$
5. $3 \times (-3)$ 13 $(-11)(-11)$ 21. $-0\ 5 \times 8\ 0 \times (-4)$
6 $-8\ \ (-7)$ 14 $(-6)(-6)$ 22. $(-3)(-6)(-7)$
7 $11(-1)$ 15 $2(-\ 3)$ 23 $(-\frac{1}{3})(-\frac{1}{3})9$
8 $-4(4)$ 16 $(\ 5)(-\ 6)$ 24 $4 \times (-\ 6) \times 7$
9. $(-6) \times (6)$ 17. $(-\frac{1}{4})(-\frac{1}{5})$ 25 $6 \times \frac{1}{2} \times \frac{1}{3}$
10 $8(-9)$ 18 $(-\frac{1}{5})(\frac{1}{4})$ 26 $(-\frac{3}{5})(5)(-\frac{1}{3})$

Division of Positive and Negative Numbers[A]

In division the number that does the dividing is the <u>divisor</u> and the number that is being divided is the <u>dividend</u> The result of the division is the <u>quotient</u>

The signs denoting division are the division sign ($\div$), the ratio sign (:), and the fraction bar. Thus $8 \div 2$, $8 : 2$, and $\frac{8}{2}$ all mean "8 divided by 2."

Law of Signs for Division

1. The quotient of two numbers with <u>like</u> signs is a positive number.
2. The quotient of two numbers with <u>unlike</u> signs is a negative number.

Examples: $10 \div 2 = 5$ $\qquad$ $(-15) \div 3 = -5$
$(-8) \div (-4) = 2$

EXERCISES [A]

Divide:

1. $10 \div 5$	6. $7 \div (-1)$	11. $8 \div (0.2)$
2. $(-12) \div (-3)$	7. $-9 \div 3$	12. $10 \div (-20)$
3. $14 \div (-7)$	8. $-10 \div 10$	13. $1.8 \div (-0.9)$
4. $-20 : 5$	9. $-15 \div 1$	14. $-1.2 \div (0.2)$
5. $18 : 9$	10. $-15 \div (-1)$	15. $1 \div 0.5$
16. $\frac{20}{-4} = ?$	19. $\frac{8}{-\frac{1}{2}} = ?$	22. $\frac{45}{-90} = ?$
17. $\frac{-80}{40} = ?$	20. $\frac{-\frac{1}{2}}{-\frac{1}{4}} = ?$	23. $\frac{-13.5}{3} = ?$
18. $\frac{-100}{-10} = ?$	21. $\frac{-\frac{2}{3}}{3} = ?$	24. $\frac{-45}{-30} = ?$

Other Numbers Belonging to Our System[A]

As you proceed through this second course in algebra you will find other numbers which have been added to our number system. Each was added in response to some need—perhaps to explain a mathematical phenomenon or to help us deal with some problem of our daily lives. In each case the new number had to be so constructed that it obeyed all the rules and laws of the already existing system.

Representing Numbers with Letters[A]

Today we often use letters to represent numbers. Letters are not used to represent a new kind of number which is to be fitted into a

particular spot of the number system as fractions and negative numbers were fitted in Instead, letters are used as supplementary symbols for already existing numbers A letter may represent a whole number, a fraction, a negative number, other numbers which you have not yet studied, or it may represent a combination of several kinds

When we write $n + 2 = 5$, the n represents only the number 3, but when we write $n^2 = 9$, the n represents either $+3$ or -3, because $(+3)(+3) = 9$ and $(-3)(-3) = 9$ In the formula for the perimeter of a square, $P = 4s$, the letter s represents any possible length of a side of a square That can be a whole number, a fraction, a mixed number, or other numbers which you have not yet studied, but it cannot represent a negative number because negative numbers as lengths for the sides of a square are meaningless to us

Words You Should Know [A]

The use of letters for numbers makes it necessary for you to know the meaning of several new words

Letters which are used for numbers are sometimes called literal numbers The expression is somewhat misleading, however, since literal numbers are not numbers, they are only symbols for number ideas Each of us has a mental concept of the meaning of one, of two, of ten, and of each of the other numbers Literal numbers represent those ideas just as do the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and combinations of them

The factors of a given number are the numbers which, when they are multiplied together, form the given number Thus 2 and 5 are the factors of 10, and 2, 3, and 7 are the factors of 42

In a product any factor is called the coefficient of the remaining factors For example, in the product $6abc$, 6 is the numerical coefficient of abc, $6ab$ is the coefficient of c, and $6bc$ is the coefficient of a

Any algebraic expression like 4, $-6ab$, or x^2 is called a term When an algebraic expression is made up of certain parts connected by $+$ and $-$ signs, these parts with the signs immediately preceding them are called the terms of the expression The algebraic expression $4 - 6ab + x^2$ has three terms, which are $+4$, $-6ab$, and $+x^2$

An algebraic expression of one term is called a monomial, one of two terms is called a binomial, and one of three terms is called a trinomial An algebraic expression of two or more terms is called a polynomial By their definitions binomials and trinomials are polynomials

Examples

Monomials: a, mn, $6x^2y$, and -4

Binomials: $+3x-2$, $-x+7$, $mn+ac$, and $6-y^2$

Trinomials: m^3-mn+n^2 and $-x^2+7x+2$

Like or similar terms are those that have the same literal factors. Thus the terms $4x^2$ and $5x^2$ are similar, and the terms $5ab^2c$ and $2ab^2c$ are similar.

Unlike terms are those that do not have the same literal factors. Thus the terms $5m$ and $2n$ are unlike, and the terms $2x^2y$ and $3xy$ are unlike.

An exponent is a small figure or letter that is written above and to the right of a number, which is called the base. In the expression x^3 the base is x and the exponent is 3, and in the expression 10^4 the base is 10 and the exponent is 4. A positive integral exponent tells how many times the base is used as a factor. For example, x^4 means $x \cdot x \cdot x \cdot x$, and 7^3 means $7 \cdot 7 \cdot 7$.

A power of a number is the product or indicated product when the number is used several times as a factor. The second power of a number is usually called the square of the number, and the third power of a number is usually called the cube of the number. Thus 9, or 3^2, is the square of 3, and 125, or 5^3, is the cube of 5.

Order of Fundamental Operations[A]

If all members of your class should simplify the expression $12 + 6 \div 2$, it is doubtful that all would obtain the same result. Some might obtain 9 as the answer, whereas others would get 15.

It is difficult to give any concise set of rules for the order of operations in such expressions, but mathematicians have agreed that all multiplications and divisions are to be performed before the additions and subtractions. The following directions will be helpful in simplifying algebraic expressions.

Rule for Order of Operations

1. First simplify all expressions which are included by symbols of grouping and find the indicated powers.
2. Perform all multiplications and divisions in order from left to right.
3. Perform all additions and subtractions.

Example 1 Simplify $3 \times 4 - 10 \div 2$

Solution
$$\begin{aligned} & 3 \times 4 - 10 \div 2 \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

Example 2 Simplify $2 \times 5^2 - 18 \div 6$

Solution
$$\begin{aligned} & 2 \times 5^2 - 18 \div 6 \\ &= 2 \times 25 - 18 \div 6 \\ &= 50 - 3 \\ &= 47 \end{aligned}$$

Example 3 Simplify $(8 + 10 \div 5)^2$

Solution $(8 + 10 \div 5)^2 = (8 + 2)^2 = 10^2 = 100$

In this example the entire expression within the parentheses is to be squared. The expression can be squared more easily when it is simplified.

Example 4 Find the value of $\dfrac{5 \times 3 - 40 \div 8}{4}$

Solution We first simplify the numerator, since the fraction bar acts as a symbol of grouping.

$$\frac{5 \times 3 - 40 \div 8}{4} = \frac{15 - 5}{4} = \frac{10}{4} = 2\tfrac{1}{2}$$

[A] **EXERCISES**

Simplify

1 $4 \times 2 + 5 \times 6$

2 $8 \times 7 - 6 \times 5$

3 $(16 \div 2) \div 8$

4 $3 \times 4^2 - 30$

5 $8 + 8 \div 8$

6 $(8 + 8) \div 8$

7 $8 + (8 \div 8)$

8 $2 \times 5 \times 4 - 20 \div 5$

9 $60 \div 4 \times 3$

10 $60 \div (4 \times 3)$

11 $(10 + 7 - 8) \div 5$

12 $\frac{2}{7}(6 + 8) \div 4$

13 $14 - (80 \div 5) + 2$

14 $7 \times 2 \div 2 + 6$

15 $\dfrac{10 - 4 + 6}{3}$

16 $3\frac{1}{7}(14)^2 16 - 10$

17 $\dfrac{7 \times 6 - 15}{10}$

18 $420 \div 14 - 35 \div 7$

19 $3 \times 1.25^2 - 4.2 \div 2$

Combining Like (Similar) Terms [A]

Like terms can be added and subtracted. For example, just as 2 pounds + 5 pounds = 7 pounds, so $2p + 5p = 7p$.

[A]

Combine like terms

1 $7a + (-3a)$

2 $x^2 - 3x^2$

3 $-ab - 4ab$

4 $7c - 9c$

5 $2x + (-3x)$

6 $-5c - c$

7 $10y - (-5y)$

8 $2b - (+3b)$

9 $2b + (-b) - 3b$

10 $5h + 2h - 3h$

11 $7k - 9k - k$

12 $-5t - (-4t)$

13 $4y - y - 6y$

14 $\frac{1}{2}x - (-\frac{1}{4}x)$

15 $\frac{2}{3}p - \frac{1}{3}p + p$

16 $x^2 - 6x^2 - \frac{1}{3}x^2$

Example Simplify $2x^2 - xy + y^2 + 2y^2 - x^2 - 3xy$

Solution $2x^2$ and $-x^2$ make x^2 $-xy$ and $-3xy$ make $-4xy$ and y^2 and $2y^2$ make $3y^2$ Then

$$2x^2 - xy + y^2 + 2y^2 - x^2 - 3xy = x^2 - 4xy + 3y^2$$

EXERCISES

[A]

Simplify

1 $a - b + c - 3a$

2 $c^2 - c + 6 - 5c$

3 $10x - 3y + z - z$

4 $a + b - 2a - 3b$

5 $x^2 - xy + y^2 - 3xy$

6 $c^2 - 4cd - d^2 + 4cd + d^2$

7 $p^2 - 4p + 2 - p^2 + 4p - 2$

8 $h + k - m - k - 5m$

9 $y^2 - 1 + y - 6 - 2y^2$

10 $x^2y - 3xy^2 - 4xy^2 - xy^2$

Addition of Polynomials [A]

The expression 2 yards + 1 foot + 6 inches is added to the expression 3 yards + 1 foot + 4 inches as shown Notice that like terms are in columns All polynomials are added in the same way

3 yd	+ 1 ft	+ 4 in
2 yd	+ 1 ft	+ 6 in
5 yd	+ 2 ft	+ 10 in

Rule for Adding Polynomials

1 Write the polynomials in column form
with like terms under one another

2 Add each column
and connect the sums with their proper signs

Example Add $3x^2 - y^2 + xy$, $2x^2 + 3xy - 4y^2$, and $2y^2 - 6xy - x^2$

Solution. The polynomials should be arranged according to the descending (or ascending) powers of one of the letters. In this solution they will be arranged in descending powers of x. We shall check the *addition* by letting $x = 2$ and $y = 3$

$$\begin{array}{rcl} \textbf{Solution} & & \textsc{check} \\ 3x^2 + \ \ xy - \ \ y^2 & = & 12 + 6 - 9 = \quad 9 \\ 2x^2 + 3xy - 4y^2 & = & 8 + 18 - 36 = -10 \\ -x^2 - 6xy + 2y^2 & = & -4 - 36 + 18 = -22 \\ \hline 4x^2 - 2xy - 3y^2 & & 16 - 12 - 27 = -23 \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} = -23$$

NOTE Though theoretically any numbers could represent x and y in the check, most mathematicians hesitate to use 0 or 1. A mistake could be overlooked when 0 is substituted, since 0 times any number is 0. When 1 is used, an expression such as $3x^2$ has the same value as either $3x$ or 3

EXERCISES

[A]

Find the sums

1
$$\begin{array}{r} 2x^2 - 3xy + y^2 \\ -4x^2 - 6xy - y^2 \\ x^2 + \ \ xy - y^2 \\ \hline \end{array}$$

2
$$\begin{array}{r} a - 3b + 4c \\ -7a \qquad\quad - 6c \\ 5b - \ \ c \\ \hline \end{array}$$

3. $x^3 - 4x^2 - 7x + 6$, $2x^3 - 5x + 2$, and $-5x^3 + 6x^2 - 8$

4. $x^3 - y^3$, $2x^3 - x^2y - 2y^3$, $x^3 - 6y^3$, and $5x^2y - xy^2 - y^3$

5. $m^2 - mn + n^2$, $4n^2 + m^2 - 3mn$, and $5m^2 + n^2$

6. $4a^3 - 5b^3 + 2a$, $6a^3 - 5a$, and $7a^3 + 9b^3 - 11a$

7. $\frac{1}{3}x + \frac{3}{4}y - \frac{2}{5}z$, $\frac{1}{4}x + y - \frac{1}{4}z$, and $\frac{1}{2}x - \frac{3}{8}y - \frac{1}{10}z$

8. $\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c$, $-\frac{2}{3}a - \frac{1}{5}b + \frac{1}{3}c$, and $a - b + c$

9. $0.7x^2 - 0.3y^2 - 0.7$, $1.2x^2 - 3.5y^2 - 0.3$, and $x^2 + y^2 - 1$

10. $1.17m - 2.15n + 1.31p$ and $4.52m - 5.65n - 4.91p$

Subtraction of Polynomials[A]

You know that 2 pounds and 7 ounces can be subtracted from 10 pounds and 11 ounces, giving 8 pounds and 4 ounces, as shown

10 pounds	+	11 ounces
2 pounds	+	7 ounces
8 pounds	+	4 ounces

Rule for Subtracting One Polynomial from Another

1 Write the polynomials in column form with like terms under one another

2 Mentally change the signs of the terms of the subtrahend and proceed as in addition

[A]

EXERCISES

Subtract

1 $$\begin{array}{r} 2x - 3y + 6z \\ \underline{4x - 5y + 2z} \end{array}$$

2 $$\begin{array}{r} -3m + 4n + 7 \\ \underline{4m - n + 6} \end{array}$$

3 $$\begin{array}{rrrr} x^3 & & & -y^3 \\ \hline -2x^3 & +x^2y & -xy^2 & +2y^3 \end{array}$$

4 $$\begin{array}{rrrr} -2a & +7b & -c & \\ \hline & -3b & +4c & -8d \end{array}$$

5 Subtract $x^2 - 6x + 5$ from $2x^2 - 4x + 3$

6 From $a^2 + ab + b^2$ take $a^2 - ab - b^2$

7 What must be added to $2x - 6$ to make $x^2 + 3x + 2$?

8 The subtrahend is $x^2 + 7xy + y^2$ and the minuend is $3x^2 + 6xy$ What is the difference?

9 By how much does $a^3 - a$ exceed $a^2 + a - 7$?

10 By how much does $x - 50$ exceed $x^2 + 50$?

11 From the sum of $2a + b$ and $3a - 4b$ take $5a + 2b$

12 From $x^2 - y^2$ subtract the sum of $x^2 + xy - y^2$ and $x^2 - xy - y^2$

13 The sides of a triangle are $5a + b$ $2a - 3b$, and $a - 9b$ Find its perimeter

14 The perimeter of a triangle is $5x^2 - x + 1$ If one side is $x^2 + 2x + 3$ and another is $x^2 - 2x + 4$, what is the third side?

15 One side of a rectangle is x and another side is $2x - 5$ Find the perimeter

Multiplying Monomials [A]

Since $x^3 = xxx$ and $x^4 = xxxx$, $x^3x^4 = xxx\,xxxx = x^7$ Similarly, $a^6a^5 = aaaaaa\ aaaaa = a^{11}$ These two examples illustrate the fact that the product of two numbers having the same base is the number with the same base and an exponent which is the sum of the exponents *of the two numbers*

Law of Exponents for Multiplication

The exponent of a number in a product
is equal to the sum of the exponents of that number in the factors

$$x^a x^b = x^{a+b}$$

Let us multiply $-3a^2b$ by $4ab^3$, $(-3a^2b)(4ab^3) = -3a^2b\ 4ab^3$ By the commutative law of multiplication this product can be written $-3\ \ 4\ \ a^2\ \ a\ \ b\ \ b^3$ By the associative law of multiplication this product becomes $-12a^3b^4$

Rule for Multiplying One Monomial by Another

1 Find the product of their numerical coefficients
to obtain the numerical coefficient of the product
2 Multiply the literal factors of the multiplicand
by the literal factors of the multiplier
to obtain the literal part of the product

Example 1 Multiply $7x^2y$ by $-3xy$ $-12a^3b^4$
Solution $(7x^2y)(-3xy) = -21x^3y^2$

Example 2 Find the product $(2ab)(-3a^2)(4a^3b)$
Solution $(2ab)(-3a^2)(4a^3b) = (-6a^3b)(4a^3b) = -24a^6b^2$

Example 3. $(2x^m)(-4x^m) = -8x^{2m}$

(A) **EXERCISES**

Find the products

1 $x^4\ \ x^3$	8 $(-2c)(-4c)$	15 $(ab^2)(-5ab)$
2 $y^3\ \ y^5$	9. $x^6(3x)$	16 $(-2xy)(-3xy^4)$
3 $c^8\ \ c^6$	10 $10^2\ \ 10^3$	17 $\frac{3}{4}(8k^2)$
4. $c\ \ c$	11 $(-ab)(ab^2)$	18. $-\frac{2}{3}(15c^2d)$
5 $-6(2x)$	12 $4m(-3n)$	19 $(3a^4)(5a)$
6 $7(-5cd)$	13 $(ab)(-ab)$	20 $(-\frac{2}{3}x)(-\frac{3}{4}x)$
7. $(-3x)(4x)$	14 $(-y)(y^4)$	21. $(\frac{1}{5}p)(\frac{1}{4}q)$

22. $2x \cdot 3x \cdot x^2$ 24 $(-3c)(-3c)(-3c)$
23 $(-6)(m^2)(mn)$ 25. $(2ab)(-bc)(-5ab^2)$

26 $c^x \cdot c^y$ 27. $a^c\ \ a^{2c}$ 28. $x\ \ x^n$

29 $x^a \cdot x^a$ 31 $x^{m+1} \cdot x^2$ 33 $2p^c \cdot 4p^c$

30 $h^{a+1} \cdot h^{a-1}$ 32 $-2x^{m+1} \cdot 3x^{1-m}$ 34 $7x^n(-3x^{1-n})$

Division of Monomials [4]

Since $x^3 \cdot x^2 = x^5$ then $x^5 \div x^2 = x^3$

Also $\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x^3$

This example illustrates the following law

Law of Exponents for Division

The exponent of any number in a quotient is equal to its exponent in the dividend minus its exponent in the divisor

$$\frac{x^a}{x^b} = x^{a-b}$$

Let us divide $42\,x^3y^2z$ by $-7\,xyz$

$$\frac{42\,x^3y^2z}{-7\,xyz} = \frac{42}{-7} \times \frac{x^3}{x} \times \frac{y^2}{y} \times \frac{z}{z} = -6\,x^2y\,1 = -6\,x^2y$$

Rule for Dividing One Monomial by Another

1 Find the quotient of the numerical coefficients to obtain the coefficient of the quotient
2 Divide the literal factors of the dividend by the literal factors of the divisor to obtain the literal factors of the quotient

Example 1 Divide $-18\,a^4bc$ by $-3\,ac$

Solution $\frac{-18\,a^4bc}{-3\,ac} = 6\,a^3b$

Remember these facts.

1 Any number divided by itself equals 1
2 Division by zero is impossible

PROOF Let x and y represent any two numbers except zero

Suppose $\frac{x}{0} = y$

Then $y \times 0 = x$

But $y \times 0 = 0$

Then $x = 0$ which is impossible

Do you see that in all divisions the letters must have values such that the divisors are not zero?

subtract exponents

[A] **EXERCISES**

Find the quotients

$x^7 \div x$

$a^3 \div a^3$

$c^8 \div c^2$

$-x^{10} \div x^9$

$x^{10} \div x^{10}$

$c^{10} \div c^8$

$10^3 \div 10$

$10^4 \div 10^2$

9 $8c \div (4c)$

10 $m^3n \div (\frac{1}{2}m^2)$

11 $(-x^2y) \div -(\frac{1}{3}xy)$

12 $(-15c^2) \div (-3)$

13 $(-12h^3) \div (4h)$

14 $\frac{1}{4}h \div (-\frac{1}{2}h)$

15 $(0.1c^8) \div (0.5c^6)$

16 $\dfrac{-x^2y}{x}$

17 $\dfrac{-20c^2t}{-4t}$

18 $\dfrac{9r^2t}{-3rt}$

19 $\dfrac{-24abc}{ab}$

20 $\dfrac{-16p^2}{-8p}$

21 $\dfrac{32a^4b}{-4a^3b}$

Example 2 $\dfrac{4x^m}{-x} = -4x^{m-1}$

EXERCISES

Divide

1 $x^m \div x^n$

2 $a^3 \div a^b$

3 $\dfrac{-10x^my^n}{-5x^my}$

4 $(8y^n) \div (-2y^{n-1})$

5 $4c^{n+1} \div c^{n-1}$

6 $\dfrac{-4x^{m+n}}{x^{m-n}}$

7 $x^{m+3} \div x^{m+1}$

8 $(-6a^hb^k) \div (3a^{-h}b^{k-1})$

9 $\dfrac{-14x^{2m}y^{3c}}{7x^{1-m}y^{-3c}}$

Multiplying Polynomials by Monomials[A]

Rule for Multiplying a Polynomial by a Monomial
Multiply each term of the polynomial by the monomial

Example $4x^2(2x^2 - 5x + 3) = 8x^4 - 20x^3 + 12x^2$

Remember that a product of a polynomial and a monomial has the same number of terms as the polynomial

[A] **EXERCISES**

1 $4(a+b)$

2 $-6(x-y)$

3 $x(2x-7)$

4 $-2y(y+6)$

5 $c^2(2c-1)$

6 $-x^2(x-y)$

7. $ab(b-a)$
8. $-xy(x^2-y^2)$
9. $m^2(n^2-2m+1)$
10. $2xy(x^2-3xy-5y^2)$
11. $-3cd(2c^2+cd-d^2)$
12. $3r^2(r^2-4rs+s^2)$
13. $-4s^2(s-4rs+r)$
14. $-1(r-s+t)$
15. $2\pi r(r+h)$
16. $-st(s^2-2st+t^2)$

17. Find the area of a floor that is $4x$ feet wide and $(7x-1)$ feet long

18. Find the area of a triangle whose base is $3m-2n$ and whose altitude is $6mn$

19. $x^n(x^{2n}-x^n+1)$
20. $y^a(y^{2a}-3y^a+4)$
21. $x^2(x^m+3x^{m-2}-7)$
22. $-y^3(y^n-2y^p-6)$
23. $xy(x^my^n-x^{m-1}y^{n+1})$
24. $-2x^{a+1}(3x^{a-1}-2x^{1-a})$

Multiplying Polynomials by Polynomials [A]

Rule for Multiplying One Polynomial by Another

1. Multiply each term of the multiplicand by each term of the multiplier
2. If possible, combine the partial products

Example Multiply $4a^2+3b^2-5ab$ by a^2-2ab

Solution The multiplicand and multiplier should be arranged in descending (or ascending) powers of the same letter We shall arrange them in descending powers of a

$$\begin{array}{rrrrl}
4a^2 & -\ 5ab & +\ 3b^2 & & \text{multiplicand} \\
a^2 & -\ 2ab & & & \text{multiplier} \\
\hline
4a^4 & -\ 5a^3b & +\ 3a^2b^2 & & \\
 & -\ 8a^3b & +\ 10a^2b^2 & -\ 6ab^3 & \\
\hline
4a^4 & -\ 13a^3b & +\ 13a^2b^2 & -\ 6ab^3 & \text{product}
\end{array}$$

The solution can be checked by going over the work or by substituting values for a and b

EXERCISES [A]

1. $(2x+1)(x+3)$
2. $(3c+1)(c+2)$
3. $(x-5)(x-5)$
4. $(ab-3)(ab+3)$
5. $(xy+7)(xy-4)$
6. $(a^2-b^2)(a^2+b^2)$

7. $(x^2 - 2y)(x^2 - 5y)$
8. $(y^2 + 2y + 1)(y + 1)$
9. $(x - y)(x^2 + xy + y^2)$
10. $(a + b)(a^2 - ab + b^2)$
11. $(a^2 + a - 1)(a^2 - a + 1)$
12. $(m^2 + n^2 - mn)(m^2 + mn + n^2)$
13. $(x^2 - 5x + 4)(x^2 - x + 2)$
14. $(1 - 4x + x^2)(1 - x)$

15. Find the volume of a rectangular solid whose dimensions are $2a + 1$, $3a - 5$, and $a - 2$

16. $(a - b)^3 = (a - b)(a - b)(a - b) = ?$

17. $(c - d)^3$
18. $(x + y)^3$
19. $(c^2 - d)^3$
20. $(a^x - b^y)(a^x + b^y)$
21. $(y^{2a} + y^a + 1)(y^a + 1)$
22. $(x^a + y^b + z^c)^2$

REVIEW EXERCISES [A]

If $A = x^2 - xy + y^2$, $B = 3x - 4y$, and $C = 2x + y$, find the value of

1. $A + B$
2. $C - B$
3. $2B + C$
4. BC
5. AC
6. AB
7. $A - BC$
8. $A + B - C$
9. $B^2 - C^2$

Dividing Polynomials by Monomials [A]

Rule for Dividing a Polynomial by a Monomial
Divide each term of the polynomial by the monomial

Example 1 Divide $5x^3y - 15x^2y^2 + 5xy$ by $5xy$

Solution $\dfrac{5x^3y - 15x^2y^2 + 5xy}{5xy} = x^2 - 3xy + 1$

Note that the quotient has the same number of terms as the dividend

Chris

BEL

Example 2. $\dfrac{4x^{2m} - 6x^{m+2} - 8x^m}{2x^m} = 2x^m - 3x^2 - 4$

EXERCISES [A]

Divide

1. $(5a^2 - 20) \div 5$
2. $(18x - 27) \div (-9)$
3. $(8x - 12y) \div (-1)$
4. $(4c - 12d) \div (-4)$
5. $(6a^3 - 12b^2) \div 3$
6. $(x^4 - x^3 + x^2) \div x$
7. $(a^4 - a^2 - a) \div (-a)$
8. $(10x^3 - 12x^2 + 2x) \div (2x)$

9 $(5a^3b + 2a^2b^2) \div (-ab)$

10 $(16x^4y^3 - 24x^2y^3) \div (8x^2y^2)$

11 $(x^{2m} + x^m) \div x^m$

12 $(y^{3c} - 4y^c) \div (-y^c)$

13 $(x^{2m} - x^{3m}) \div x^{2m}$

14 $(x^c - x^{c+1}) \div x^c$

15 $(y^{n+1} - 2y^{n-1} + y^2) \div y^2$

16 $(15c^n - 10c^{n+3} - c^3) \div c^n$

Example 3 Divide $(a - b)x + 2(a - b)y + a - b$ by $a - b$

Solution The term $(a - b)x$ divided by $(a - b)$ equals x, the term $2(a - b)y$ divided by $(a - b)$ equals $2y$ and $a - b$ divided by $a - b$ equals 1 Then

$$\frac{(a-b)x + 2(a-b)y + (a-b)}{a-b} = x + 2y + 1$$

$$\frac{ax - bx + 2ay - 2by + a - b}{a - b} \quad x =$$

EXERCISES

1 Divide $3(a - b) + n(a - b)$ by $a - b$

2 Divide $(a + b)^2 + c(a + b)$ by $a + b$

3 $[x^2(x + 2y) - y^2(x + 2y)] \div (x + 2y)$

4 Divide $c^2(2c - 3) + 2c(2c - 3)$ by $2c - 3$

5. $\dfrac{a^2(x-4) - 2a(x-4) + x - 4}{x - 4} = ?$

Dividing Polynomials by Polynomials [A]

If you have forgotten how to divide one polynomial by another, use pencil and paper in doing Examples 1 and 2 following, referring to the solutions in the text only when necessary

Rule for Dividing One Polynomial by Another

1 Arrange both dividend and divisor in descending (or ascending) powers of one letter
2 Divide the first term of the dividend by the first term of the divisor
3 Multiply the entire divisor by this result and subtract the result from the dividend
4 Consider the remainder as a new dividend and repeat steps 2 and 3 as long as division is possible

Example 1 Divide $x^3 - 2x - x^2 + 8$ by $x + 2$

Solution

$$
\begin{array}{r}
x^2 - 3x + 4 \\
x+2 \,\big)\, x^3 - x^2 - 2x + 8 \\
\underline{x^3 + 2x^2 \qquad\qquad\;\;} \\
-3x^2 - 2x + 8 \\
\underline{-3x^2 - 6x \quad\;\;} \\
4x + 8 \\
\underline{4x + 8}
\end{array}
$$

CHECK

If $x = 2$, $x^3 - x^2 - 2x + 8 = 8$

If $x = 2$, $x + 2 = 4$

If $x = 2$, $x^2 - 3x + 4 = 2$

$8 \div 4 = 2$

The division can be checked by multiplying the quotient by the divisor, which should give the dividend as the product. The division can also be checked by substituting any value other than -2 for x. However, 0 and 1 are not good numbers to use.

Example 2. $(x^3 + 27) \div (x - 3)$

Solution Since the dividend contains no x^2 term and no x term, spaces should be reserved for such terms in case they are needed in the division.

$$
\begin{array}{rrl}
 & x^2 + 3x + 9 & \text{quotient} \\
\text{divisor } x-3 \,\big) & x^3 + \;\; 0 + 0 + 27 & \text{dividend} \\
 & \underline{x^3 - 3x^2 \qquad\qquad\;\;} & \\
 & 3x^2 + \;\; 0 + 27 & \\
 & \underline{3x^2 - 9x \qquad} & \\
 & 9x + 27 & \\
 & \underline{9x - 27} & \\
 & 54 & \text{remainder}
\end{array}
$$

The solution can be checked by multiplying $x^2 + 3x + 9$ by $x - 3$ and adding 54 to the product. The final result equals $x^3 + 27$. The division can also be checked by substituting any value other than 3 for x, although 0 and 1 should not be used.

(A) **EXERCISES**

Divide

1. $(x^2 + 4x + 4) \div (x + 2)$
2. $(a^2 + 6a + 9) \div (a + 3)$
3. $(m^2 - 6m + 10) \div (m - 3)$
4. $(c^2 - 8c + 9) \div (c - 4)$
5. $(y^2 - 10y + 25) \div (y - 5)$
6. $(x^2 - 17x + 60) \div (x - 12)$
7. $(4a^2 - 9) \div (2a + 3)$
8. $(9t^2 - 16) \div (3t - 4)$

9 $(m^2n^2 - 10\,mn + 24) \div (mn - 6)$

10 $(x^2y^2 - xy - 6) \div (xy + 2)$

11 $(x^6 - 13x^3 + 42) \div (x^3 - 7)$

12 $(x^8 - 3x^4 - 21) \div (x^4 + 3)$

13 $(x^3 + y^3) \div (x + y)$

14 $(x^3 - y^3) \div (x - y)$

15 $(c^3 - 27) \div (c - 3)$

16 $(h^3 - 64) \div (h + 4)$

17 $(6\,x^3 + 29\,x + 25) \div (2\,x + 5)$

18 $(a^3 - 3\,a^2b + 3\,ab^2 + b^3) \div (a - b)$

19 $(a^3 - a^2y + ay^2 - y^3) \div (a + y)$

20 $(x^4 + x^2y^2 + y^4) \div (x^2 + xy + y^2)$

[B]

21 $(x^4 + 4\,y^4) \div (x^2 - 2\,xy + 2\,y^2)$

22 $(x^6 + x^4 - x^3 + x^7 + 1) \div (x^2 + x + 1)$

23 $(2\,x^3 + x^4 - 6\,x^2 + 11\,x - 10) \div (x^2 + 2 - x)$

24 $(x^{2n} + 5\,x^n - 24) \div (x^n + 8)$

25 $(x^{2n} - x^n - 6) \div (x^n + 2)$

26 $(a^{3n} + b^{3n}) \div (a^n + b^n)$

27 $(x^{3a} - y^{3a}) \div (x^a - y^a)$

28 $(x^5 + y^5) \div (x + y)$

29 $(x^{5n} - y^{5n}) \div (x^n - y^n)$

30 $(x^6 - y^6) \div (x^2 + y^2)$

TO THE TEACHER If desired synthetic division (page 525) may be studied at this time

Symbols of Grouping [A]

The symbols for grouping algebraic expressions are the parentheses ( ), the brackets, [ ], the braces, { }, and the vinculum, or bar, ——— The line which separates the numerator and denominator of a fraction is a symbol of grouping, meaning that the whole numerator is to be divided by the whole denominator The word "parentheses" is used to include brackets, braces, and vincula (plural of "vinculum")

A quantity that is enclosed by a symbol of grouping must be treated as a whole Thus $(3\,a)^2$ means that $3\,a$ is to be squared And again, $4\,x - (3\,x + 1)$ means that $3\,x + 1$ is to be subtracted from $4\,x$

Parentheses can be removed if the indicated operations on them are performed When parentheses are preceded by a plus or minus sign, the following rules can be used

Rules for Removing Parentheses

1. When removing parentheses preceded by a plus sign, do not change the signs of the terms within.
2. When removing parentheses preceded by a minus sign, change the signs of the terms within

Example 1 Simplify $10 - 3(x + 2)$

Solution $10 - 3(x + 2)$ means that $x + 2$ is to be multiplied by 3 and that the product is to be subtracted from 10 Then

$$\begin{aligned} &10 - 3(x + 2) \\ &= 10 - (3x + 6) \\ &= 10 - 3x - 6 \\ &= -3x + 4 \end{aligned}$$

We usually shorten the work by multiplying $x + 2$ by -3 and combining like terms This solution is written

$$\begin{aligned} &10 - 3(x + 2) \\ &= 10 - 3x - 6 \\ &= -3x + 4 \end{aligned}$$

Example 2 Simplify $3(2a - b) - 4(a - 3b)$

Solution

$$\begin{aligned} &3(2a - b) - 4(a - 3b) \\ &= 6a - 3b - (4a - 12b) \\ &= 6a - 3b - 4a + 12b \\ &= 2a + 9b \end{aligned}$$

The work just shown is shortened as follows

$$\begin{aligned} &3(2a - b) - 4(a - 3b) \\ &= 6a - 3b - 4a + 12b \\ &= 2a + 9b \end{aligned}$$

Example 3 Simplify $(x^2 - 3x + 2) - (10 - 4x^2)$

Solution 1 Since $10 - 4x^2$ is to be subtracted, its signs must be changed Then

$$\begin{aligned} &(x^2 - 3x + 2) - (10 - 4x^2) \\ &= x^2 - 3x + 2 - 10 + 4x^2 \\ &= 5x^2 - 3x - 8 \end{aligned}$$

Solution 2 The value of the expression is not changed if the coefficient 1 is written before each quantity in parentheses Then

$$\begin{aligned}&(x^2 - 3x + 2) - (10 - 4x^2)\\ &= 1(x^2 - 3x + 2) - 1(10 - 4x^2)\\ &= x^2 - 3x + 2 - 10 + 4x^2\\ &= 5x^2 - 3x - 8\end{aligned}$$

EXERCISES [A]

Remove symbols of grouping and combine like terms

1 $4x + (2 - x)$

2 $5a + (2a + 6)$

3 $7c - (2c - 1)$

4 $4b + (3b + 2)$

5 $2x - 3(4 + x)$

6 $y - 5(2y - 1)$

7 $3(2x - 3) + 4(x + 2)$

8 $5(y + 1) + 2(y - 3)$

9 $x(x - 1) - 2x(x + 2)$

10 $y(2y - 3) - 2y(y + 1)$

11 $2x^2(2x - 1) - x(x + 2)$

12 $c^2(c - 3) - 3c^2(c + 4)$

13 $(2x^2 - x + 3) + (x^2 - 5x + 5)$

14 $(3x + 2y + 1) - (x - 2y + 3)$

15 $a + b + c - (2a + b - c)$

16 $5x + (3x - y) + (-y - 3x)$

17 $-(7a - 4b + 5c) - (a + 2b - c)$

18 $-(x + y - z) + (y + x - z)$

19 $8(\frac{1}{2}a - \frac{1}{4}b + \frac{1}{8}) - 6(\frac{2}{3}a - \frac{1}{6}b + \frac{1}{2})$

20 $\frac{1}{3}(6x^2 - 9x + 12) - \frac{1}{4}(4x^2 - 16x + 4)$

Enclosing Terms by Symbols of Grouping [A]

When parentheses preceded by a plus sign are removed, the signs of the terms within are not changed, and when parentheses preceded by a minus sign are removed, the signs of the terms within are changed So when terms are enclosed by parentheses, we use the following rules

Rules for Enclosing Terms in Parentheses

1 Terms can be enclosed by parentheses preceded by a plus sign without changing their signs

2 Terms can be enclosed by parentheses preceded by a minus sign if the signs of the terms are changed

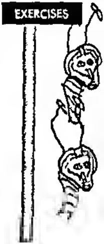
Example 1 Enclose the last two terms of $a + b - c$ by parentheses preceded by a plus sign

Solution $a + b - c = a + (b - c)$

Example 2 Enclose the last two terms of $a + b - c$ by parentheses preceded by a minus sign

Solution $a + b - c = a - (-b + c)$

Why are the signs of b and c changed in Example 2 but not in Example 1?



[A]

Enclose the last two terms of each polynomial by parentheses preceded by a plus sign

1 $m + n + p - q$	5 $c - d + e - f$
2 $m - n - p + q$	6 $x^3 - 4x^2 - 6x + 2$
3 $4 + x^2 + 2x - 3y$	7 $a^2 + b^2 + c^2 - d^2$
4 $a^2 - 2a + 6$	8 $a^2 - k^2 - b^2 - h^2$

Enclose the last two terms of each polynomial by parentheses preceded by a minus sign

9 $a - b - c - d$	13 $x - 1 - x^2 + 1$
10 $a + b - c + d$	14 $a^2 - b^2 - c^2 + d^2$
11 $x^2 - 6x + 9$	15 $m^2 - 3m - n^2 - p^2$
12 $a^2b^2 + ab - 6$	16 $c - d - m + n$

Nests of Parentheses [A]

It is often necessary in mathematics to enclose one symbol of grouping by another When one symbol of grouping is enclosed by another, either can be removed first In most cases, however, it is better to remove the innermost symbol of grouping first To avoid errors, do one operation at a time

Example Remove symbols of grouping and combine like terms $5x - [(3x + 1) - 3 - (2x - 3)]$

Solution 1

$$\begin{aligned} & 5x - [(3x + 1) - 3 - (2x - 3)] \\ &= 5x - [3x + 1 - 3 - 2x + 3] \\ &= 5x - 3x - 1 + 3 + 2x - 3 \\ &= 4x - 1 \end{aligned}$$

Solution 2 Some teachers prefer the following solution

$$
\begin{aligned}
&5x - [(3x + 1) - 3 - (2x - 3)] \\
&= 5x - [3x + 1 - 3 - 2x + 3] \\
&= 5x - [x + 1] \\
&= 5x - x - 1 \\
&= 4x - 1
\end{aligned}
$$

EXERCISES

Remove symbols of grouping and combine like terms

1 $7a - [(2a + 5) - 6 - (2a + 7)]$

2 $p + \{3r + (3p - 4r) - 1\}$

3 $x^2 - (2x - y^2) - [x^2 - (2xy - y^2)]$

4 $3y - 7 - (y - 5 - \overline{2y + 6})$

5 $2a + \{-2a - (3a + 2)\}$

6 $2y + \{3y - (y + 5)\}$

7 $5c - 2\{6c - (-c + 2)\}$

8 $m^2 - \{2m^2 + 6 - 2(m + 2 - m^2)\}$

9 $2(a - b) - \{2a - (a - b)\}$

10 $y(y - 1) - \{y^2 - 2y + y(y - 1)\}$

Powers of Numbers[A]

Although find ng powers of a number is not one of the four func mental processes it is cons dered in this chapter because it is bas upon multiplication which is one of the four processes

Since $5^3 = 5 \quad 5 \quad 5 = 125$ the third power of 5 is 125 and sir $(3m^4)^5 = (3m^4)(3m^4)(3m^4)(3m^4)(3m^4) = 243m^{20}$ the fifth power $3m^4$ is $243m^{20}$ In the expression 10^2 the *base* is 10 and the *expon* is 2 and in the expression $(2y)^3$ the base is $2y$ and the exponent is

The law of exponents for powers can be expressed

$$(a^m)^n = a^{mn}$$

Law of Signs for Powers

1 Even powers of any real number are pos t ve

2 Odd powers of any real number have the same sign as the number itself

3'

Example. $(-6\,ab^2c^3)^3 = ?$

Solution $(-6\,ab^2c^3)^3 = (-6\,ab^2c^3)(-6\,ab^2c^3)(-6\,ab^2c^3)$
$= -216\,a^3b^6c^9$

The short method of finding the third power of $-6\,ab^2c^3$ is as follows Since we are finding an odd power of a negative number, the sign of the power is minus The cube of 6 is 216, the cube of a is a^3, the cube of b^2 is b^6, and the cube of c^3 is c^9

Then $(-6\,ab^2c^3)^3 = -216\,a^3b^6c^9$

EXERCISES [A]

1 What is the meaning of x^3? of $(2\,b)^6$? of $(7\,c^2)^5$?

2 Write $(2\,n)(2\,n)(2\,n)(2\,n)$ using $2\,n$ as the base

3 Write $(5\,x)(5\,x)(5\,x)$ using 3 as the exponent

Find the indicated powers mul. exp. by power

4 $(a^2)^3$	9 $(\frac{1}{2})^3$	14 $(x^m)^3$
5 $(4\,n)^3$	10 $(\frac{2}{3})^2$	15 $(y^m)^4$
6 $(c^2)^2$	11 $(ab^2)^3$	16 $(-3\,x^2y)^4$ = 81 etc.
7 $(x^2)^4$	12 $(a^3b)^4$ $a^{12}b^4$	17 $(\frac{3}{4}\,a^2b)^3$
8 $(-4)^2$	13 $(x^{\frac{1}{2}})^4$ x^2	18 $(-\frac{1}{3}\,a^2b)^3$

19. Express as powers of 10 10, 100, 1000, 100,000

20. Express as powers of 2 8, 16, 64, 128, 256

Roots of Numbers [A]

If a number is the product of two or more equal factors, any one of these factors is a root of the number Since $9 = 3 \cdot 3$, a square root of 9 is 3 Also, since $9 = (-3) \times (-3)$, a square root of 9 is -3 Then 9 has two square roots, $+3$ and -3 Likewise, any number except zero has two square roots, which have equal absolute values but are opposite in sign The positive square root of a number is called the principal square root of the number The principal square root of a number is indicated by the radical sign ($\sqrt{\ }$) The negative square root of a number is indicated by writing the minus sign before the radical sign Thus $\sqrt{16}$ means $+4$ and $-\sqrt{16}$ means -4

If a number has three equal factors, any one of them is called the <u>cube root</u> of the number. Since $27 = 3 \times 3 \times 3$, a cube root of 27 is 3. Since $-64 = (-4)(-4)(-4)$, a cube root of -64 is -4. Any number except zero has two square roots, three cube roots, and n nth roots. The principal cube root of 64 is indicated by $\sqrt[3]{64}$, and the principal cube root of -125 is indicated by $\sqrt[3]{-125}$. In general, the <u>principal nth root</u> of a number R is indicated by $\sqrt[n]{R}$. In the expression $\sqrt[n]{R}$, n is the <u>index of the root</u> and R is the <u>radicand</u>. The index is always written unless it is 2. For example, $\sqrt{25}$ means $\sqrt[2]{25}$.

The law of exponents for roots is

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Law of Signs for Roots

1. An even root of a number may have either sign.
2. An odd root of a number has the same sign as the number.

Example Find the indicated root. $\sqrt[3]{125\,x^6}$

Solution The cube root of 125 is $+5$ and the cube root of x^6 is x^2.

Then $$\sqrt[3]{125\,x^6} = 5\,x^2$$

NOTE TO PUPIL [C] We cannot tell whether the principal square root of x^2 is $+x$ or $-x$ unless we know whether x is a positive or a negative number. For example

If $x = 2$, then $+\sqrt{x^2} = +\sqrt{2^2} = +\sqrt{4} = +2 = x$

If $x = -2$, then $+\sqrt{x^2} = +\sqrt{(-2)^2} = +\sqrt{4} = +2 = -x$

In other words, if x is a negative number, the principal square root of x^2 is not x because the principal square root must be a positive number; but if x is a positive number, the principal square root of x^2 is x. To avoid ambiguous situations the following statement will hold in this book unless otherwise specified.

If the index of a radical is an even integer, all literal numbers in the radicand, excepting those used as exponents, represent positive numbers and are such that the radicand is positive.

[A] **EXERCISES**

Find the indicated roots

1. $\sqrt{25}$	9. $\sqrt[3]{x^6}$	17. $\sqrt[5]{-32y^{15}}$	25. $\sqrt{\frac{1}{4}}$
2 $-\sqrt{49}$	10 $\sqrt[3]{y^{12}}$	18 $\sqrt[5]{a^5b^{10}}$	26 $-\sqrt{\frac{9}{25}}$
3 $\sqrt{64}$	11 $\sqrt[3]{m^{15}}$	19 $\sqrt{x^{4m}}$	27. $\sqrt{0.16}$
4 $\sqrt{x^6}$	12 $\sqrt[3]{-64y^{21}}$	20. $\sqrt{a^{6n}}$	28 $\sqrt{0.25}$
5 $\sqrt{a^{10}}$	13 $\sqrt{4a^2b^4}$	21. $\sqrt[4]{y^{2a}}$	29 $\sqrt{0.0001}$
6 $\sqrt{m^{12}}$	14 $-\sqrt{25a^{10}}$	22 $\sqrt[6]{x^{3b}}$	30 $\sqrt{0.04c^6}$
7 $\sqrt[3]{27}$	15 $\sqrt{49x^6}$	23. $\sqrt[3]{x^3y^{6a}}$	31 $\sqrt{x^{4.6}}$
8 $\sqrt[3]{-8}$	16 $\sqrt[4]{16x^8}$	24 $\sqrt[3]{c^6x^{6b}}$	32 $\sqrt[3]{m^{3.9}}$

Evaluating Algebraic Expressions [A]

The value of an algebraic expression is found by substituting for the letters their values and simplifying the resulting expression

Example 1 Find the value of $3x^3 + 5xy$ when $x = 2$ and $y = -3$

Solution
$$\begin{aligned} &3x^3 + 5xy \\ &= 3(2)^3 + 5(2)(-3) \\ &= 24 - 30 \\ &= -6 \end{aligned}$$

Example 2 Evaluate $a^2 + 2ab - b^2$ when $a = -5$ and $b = 3$

Solution
$$\begin{aligned} &a^2 + 2ab - b^2 \\ &= (-5)^2 + 2(-5)(3) - (3)^2 \\ &= 25 - 30 - 9 \\ &= -14 \end{aligned}$$

[A] **EXERCISES**

If $a = 3$, $b = -2$, $c = 2$, and $d = 0$, find the value of

1. $c^2 - 3b$	8 $c^2 - cd$	15 $7abc - a^3$
2. $2a^3 + b^3$	9 $5c^2 - 2ab$	16 $a^3b - ab^3$
3 $c^2 + bd$	10 $a^3b - 6$	17. $\frac{1}{2}(b - c - 2)$
4 $6a^3 + bc$	11 $bc + ad$	18 $(bc + a^3 - 2) \div a$
5. $5c + 5ab$	12 $3(a + b + c)$	19 $(ab)^2 + 2(bc)^2$
6 $(7b)^3 - 5d$	13 $abd - c$	20 $(4a^2b) \div (-9c)$
7 $\frac{ab - 8}{7}$	14 $\frac{a - b + c}{a + b - c}$	21. $\frac{4a^2}{c} - b^2d^2$

Checking Your Understanding of Chapter I

This first chapter has presented the basic principles of operating with positive and negative numbers and with letters used to symbolize numbers These principles are the foundation upon which all the remainder of this course will rest so it is important that you understand the ideas presented, and be able to perform with ease and accuracy the operations mentioned Make sure that you

Should you review?

1 Can add (page 8), subtract (page 9) multiply (page 10) and divide (page 11) positive and negative numbers

2 Know the recognized order in which operations should be performed (page 14)

3 Can combine like terms in a polynomial (page 15) and can add (page 16) and subtract (page 17) polynomials

4 Know the basic principles for multiplying (page 18) and dividing (page 20) monomials

5 Know how to multiply a polynomial by a monomial (page 21) and by another polynomial (page 22)

6 Know how to divide a polynomial by a monomial (page 23) and by another polynomial (page 24)

7 Know the principles for removing parentheses (page 26) and for enclosing terms in parentheses (page 28)

8 Can find indicated powers (page 30) and roots (page 31)

9 Can evaluate simple algebraic expressions such as those given in this chapter (page 33)

10 Know the meaning of the following expressions and can spell the words in them

MATHEMATICAL VOCABULARY

	PAGE		PAGE
absolute value	8	cube of a number	14
addend	9	cube root	32
associative law	6	difference	10
base of a power	14	distributive law	6
binomial	13	dividend	11
coefficient	13	divisor	11
commutative law	5	exponent	14

factor	13	principal root	31, 32
index of a root	32	product	10
like terms	14	quotient	11
literal number	13	radical sign	31
minuend	10	radicand	32
monomial	13	similar terms	14
multiplicand	10	square of a number	14
multiplier	10	square root	31
negative number	7	subtrahend	10
polynomial	13	sum	9
positive number	8	term	13
power of a number	14	unlike terms	14

CHAPTER REVIEW

[A]

1 Define minuend subtrahend multiplicand product dividend

2 What name is given to the parts of an algebraic expression?

3 How many terms has a monomial? a binomial?

4 State the law of exponents for multiplication for division

5 Complete In the expression $4x^3$ the 4 is the ? of x^3 and the 3 is the ? of x

6 Complete In the expression $4ab$ the 4 the a and the b are the ? of $4ab$

7 Simplify $16 \div 2 - 9 \div 6 + 4(1.25)$

8 Simplify $(8 + 2) - (20 \div 5)$

9 Simplify $2(x^2 - 6x - 4) - (x^2 - 3x)$

If it is given that $A = m^2 - 5mn + n^2$ $B = 2m^2 - 3n^2$ and $C = m^2 + mn - n^2$, find the value of

10 $A + B + C$	12 $A - (B + C)$	14 $3B + 2C$
11 $A - B + C$	13 $2A - 2B$	15 $C - A$

16 Perform the indicated operations

a $(-6) + (-7)$	d $10x^2 \cdot 4x^3$	g $x^{a+1} \div x^{a-1}$
b $10^7 \div 10^2$	e $2a^m \cdot (-3a)$	h $(c^3)(-3c^4)$
c $-8 - (+8)$	f $6x^{3m} \div (2x^m)$	i $(-2xy^2)(xy)$

Do as indicated

17 $(2m+1)(3m-4)$

18 $(a^2-a+1)(a+1)$

19 $(x^3-x^2+x-1)+(x-1)$

20 $(5a^2b^m-100a^3b)+(5ab)$

Remove symbols of grouping and combine like terms

21 $10-(a-1)$

22 $3x+(7-2x)$

23 $x(x-1)-2x(x-3)$

24 $(a-b)-[(b+c)-(a-c)]$

25 Enclose the last three terms of $x^4-2x^3-4x^2-x+1$ by parentheses preceded by a minus sign

26 Find the indicated powers

a $(x^3)^5$ c $(2ad)^4$ e $(10^2)^3$

b $(-2x)^3$ d $(x^m)^p$ f $(-\frac{1}{3}xy^2)^3$

27 Find the indicated roots

a $\sqrt{64}$ c $\sqrt{49x^6y^8}$ e $\sqrt{10^6}$

b $-\sqrt{25}$ d $\sqrt[3]{-8a^3b^{3n}}$ f $\sqrt[3]{10^9}$

28 If $x=-3$ and $y=4$ find the value of

a x^3+y c $\dfrac{x^2-y^2}{x-y}$ e $\frac{1}{2}(x-y)-2$

b $5x^2-3y$ d $2y^3-x$ f $(y-x)+7(x+y)$

CHAPTER TEST

[A]

1 Perform the indicated operations

a $3-6+8$ d $-6+(-7)$ g $20(-\frac{3}{5})$

b $(30)+(-6)$ e $(8a)(-4b)$ h $(-4)-(-8)$

c $8-(-2)$ f $12x^2+x$ i $3y^2 \quad 2y^3$

2 Add a^3-6a^2-a+3 a^3-4a^2+8 and $12-a+a^2$

3 From $3x-7y+z$ take $7x-4y+6z$

4 Multiply y^3-2y-3 by $2y-1$

5 Divide $6x^3-x-4-11x^2$ by $2x-3$

6 Simplify $4(x-1)+2(3-x)-5(x-2)$

7 Divide $20m^3-15m^2-5m$ by $5m$

8. If $x = 3$ and $y = -2$, find the value of $x^2y - 20$

9 From the sum of $a^2 - a + 7$ and $3a^2 - 2a - 9$ take $-a^2 + 6a - 2$

10 Simplify $12 \div 2 - 8 \div 4$

11. Divide $y^3 - 8$ by $y - 2$

12 The length of a rectangle is $3a - b$ and its width is $a + b$ Find its perimeter and area

13 Find the indicated powers

a. $(-2y)^3$	c $(-ac^2)^4$	e $(x^m)^n$
b $(x^2y)^2$	d $(10^3)^2$	f $(\frac{1}{3}a^2b)^2$

14 Find the indicated roots

a. $\sqrt{49}$	c $\sqrt{x^6}$	e $\sqrt[5]{y^{10}}$
b $\sqrt[3]{-27}$	d. $-\sqrt{36}$	f $\sqrt[4]{16x^{12}}$

15 Square $x^2 + x - 1$

16 Cube $x - y$

17 Divide $x^5 + y^5$ by $x + y$

18 Enclose the last two terms of $2a + b - c$ in parentheses preceded by a minus sign

19 Subtract $a^2 - 4ab + b^2$ from $6ab$

20 Find the value of $(4ab)^2 - 40$ when $a = -3$ and $b = 4$

CHAPTER 2

First-Degree Equations in One Unknown

In this chapter you study some of the basic principles of equations

In algebra we arrange numbers in patterns and study the relationships we find. The equation is one of the most important of these patterns for it gives algebra amazing power to solve problems. Without equations much of our knowledge in science, in engineering, and in industry would have been impossible. You will want to become an artist in understanding and using this important concept.

The Meaning of "Equation" [A]

An equation is a statement that two quantities are equal. The two quantities are called members of the equation. The statement $2x - 3 = x + 4$ is an equation because it says that the quantity represented by the expression $2x - 3$ is equal to the quantity represented by the expression $x + 4$.

It is easily seen, however, that $2x - 3$ does not equal $x + 4$ for all values of x. If, for example, we let x represent 2, the left member $2x - 3$ becomes $2 \cdot 2 - 3$, or 1, and the right member $x + 4$ becomes $2 + 4$, or 6. Since 1 does not equal 6, we see that the statement is not true when $x = 2$. It is just as easily seen that the statement is true when x is replaced by 7, for then the left member becomes $2 \cdot 7 - 3$, or 11, and the right member becomes $7 + 4$, or 11.

An equation which is true for some values of the letters it contains, but is not true for others, is called a conditional equation. A conditional equation is true only on condition that its letters represent particular values.

Certain other equations are true for any values of the letters they contain. Thus, $3(x + 4) = 3x + 12$ is true when $x = 1$, when $x = 2$, when $x = 3$, or any other number. Try substituting various values for x to convince yourself that the statement is always true. When a statement expressing equality is true for any value of the letters it contains (if it contains letters), it is called an identical equation.

Some identical equations contain no letters. The equation $2 + 5 = 3 + 4$ is such an identical equation. When we replace the letters of a conditional equation by values which will make the equation true, we change the conditional equation to an identical equation.

Identical equations are usually referred to as identities, and conditional equations as equations. In this book the term *equation* will be used for a conditional equation, and the term *identity* will be used for an identical equation.

[A]

EXERCISES Which of the following are identities and which equations?

1. $x + 7 = 11$
2. $3x + 2 = x + 6$
3. $2 \cdot 3 - 1 = 5$
4. $3(x + 2) = 6 + 3x$
5. $x(x - 6) = 5$
6. $8 \cdot 0 = 0$
7. $(r + s)(r - s) = r^2 - s^2$
8. $c^x \cdot c^y = c^{x+y}$
9. $\frac{2}{x} + \frac{6}{x} = \frac{8}{x}$ BE CAREFUL ON THIS ONE

The Root of an Equation [A]

The process of finding the numbers represented by the letters of an equation is called solving the equation, and the set of numbers found is called the solution. The solution satisfies the equation, that is, changes it to an identity. In the equation $y = x + 2$, the set of numbers, $y = 5$, $x = 3$, makes up a solution. If the equation contains only one letter, we call the solution a root of the equation. In the equation $x + 4 = 6$, the solution $x = 2$ is a root. Until we have solved an equation, its letters represent unknowns.

A first degree equation in one unknown is an equation in which the unknown has only the exponent 1 after the equation has been cleared of fractions, parentheses, and radical signs. You will find that a first degree equation in one unknown has only one root.

Solving an Equation [A]

An equation relates the unknown to known numbers by addition, subtraction, multiplication, or division. For example, the equation $3n + 4 = 25$ relates the unknown n to 3 by multiplication, and then ties the product to 4 by addition. If we systematically undo the ties which bind the unknown to the other numbers, being careful always to preserve the equality of the two members, we can obtain a simple equation with only the unknown in one member and only a known

number in the other In the equation $3n + 4 = 25$, we may proceed as follows

Solution $3n + 4 = 25$

STEP 1 If we subtract 4 from each member, we obtain

$$3n = 21$$

STEP 2 If we then divide each member by 3, we obtain

$$n = 7$$

Did you notice that in the words underlined at the foot of page 40 we proposed to do two things (1) to undo the ties that hold the unknown to other numbers, and (2) to keep the two members equal at all times?

To accomplish the first of these tasks, that is, to undo the ties, we used the idea of inverse processes We know that an addition will undo a subtraction, and vice versa, and we know that a multiplication will undo a division, and vice versa In Step 1 of the solution we *subtracted 4* because, in the original equation, *4 had been added* to the term containing the unknown In Step 2 of the solution we *divided by 3* because, in the equation $3n = 21$, the unknown was *multiplied by 3*

To accomplish the second task, that is, to keep the two members of each succeeding equation equal, we made use of statements 2 and 4 of the following assumptions These assumptions (sometimes called axioms) are statements which everyone believes true, even though we cannot prove them

Our Basic Assumptions about Equation Solving

1 If *the same number is added to each member of an equation*, the members of the resulting equation are equal
2 If the same number is subtracted from each member of an equation, the members of the resulting equation are equal
3 If each member of an equation is multiplied by the same number, the members of the resulting equation are equal
4 If each member of an equation is divided by the same number (except 0), the members of the resulting equation are equal
5 A quantity may be substituted for its equal in any expression

Proof in Equation Solving [A]

When we arrive at the statement $n = 7$, in solving the equation $3n + 4 = 25$ on the preceding page, we have only shown that if the equation has a root then 7 is that root. We are still faced with the task of showing that 7 is a root. To do so, we make use of Assumption 5 by substituting 7 for n in the equation thus:

PROOF	$3n + 4 = 25$
Does	$3 \cdot 7 + 4 = 25?$
Does	$21 + 4 = 25?$
Does	$25 = 25?$ Yes

Since the two members of $3n + 4 = 25$ are equal when $n = 7$, we know that 7 is a root.

Remember Proof is a vital part of every complete solution of an equation.

Practice in Equation Solving [A]

The following statements summarize the steps in solving an equation. Study the statements and the examples which follow them before you try the exercises in this section.

To Solve a First Degree Equation in One Unknown

1. If the equation contains fractions,
   reduce each fraction to lowest terms
   and multiply both members by the L.C.D. of the fractions.
2. If there are parentheses in the equation,
   remove them.
3. If there are like terms in either member,
   combine the terms.
4. Use the idea of inverse processes
   and the basic assumptions concerning equation solving
   to get the unknown alone in one member
   and a known number alone in the other.
5. Prove that the value found for the unknown
   is a root of the original equation.

Example 1 Solve $\frac{1}{3}x = \frac{1}{4}x - 2$

Solution 1

	$\frac{1}{3}x = \frac{1}{4}x - 2$
*M_{12}	$4x = 3x - 24$
S_{3x}	$x = -24$

PROOF Does $\frac{1}{3}(-24) = \frac{1}{4}(-24) - 2$?
Does $-8 = -6 - 2$?
Does $-8 = -8$? Yes

You may prefer this solution which uses an extra step

Solution 2

	$\frac{1}{3}x = \frac{1}{4}x - 2$
M_{12}	$4x = 3x - 24$
S_{3x}	$4x - 3x = -24$
*C T	$x = -24$

PROOF Same as for Solution 1

Example 2 Solve $2(y+1) = 4 - (3y - 8)$

Solution 1

	$2(y+1) = 4 - (3y - 8)$
	$2y + 2 = 4 - 3y + 8$
C T	$2y + 2 = 12 - 3y$
S_2	$2y = 10 - 3y$
A_{3y}	$5y = 10$
D_5	$y = 2$

PROOF
Does $2(2+1) = 4 - (3 \cdot 2 - 8)$?
Does $2(3) = 4 - (-2)$?
Does $6 = 6$? Yes

Solution 2

	$2(y+1) = 4 - (3y - 8)$
	$2y + 2 = 4 - 3y + 8$
C T	$2y + 2 = 12 - 3y$
S_2	$2y = 12 - 2 - 3y$
C T	$2y = 10 - 3y$
A_{3y}	$2y + 3y = 10$
C T	$5y = 10$
D_5	$y = 2$

PROOF Same as for Solution 1

If you prefer, the S_2 and A_{3y} steps may be combined. The two solutions would then appear as follows

Solve $2(y+1) = 4 - (3y - 8)$

Solution 1

	$2(y+1) = 4 - (3y - 8)$
	$2y + 2 = 4 - 3y + 8$
C T	$2y + 2 = 12 - 3y$
S_2 A_{3y}	$5y = 10$
D_5	$y = 2$

PROOF As above

Solution 2

	$2(y+1) = 4 - (3y - 8)$
	$2y + 2 = 4 - 3y + 8$
C T	$2y + 2 = 12 - 3y$
S_2 A_{3y}	$2y + 3y = 12 - 2$
C T	$5y = 10$
D_5	$y = 2$

PROOF As above

*M_{12} means multiplying both members of the equation by 12. A, S and D will be used for adding, subtracting and dividing respectively. C T means combining terms in each member.

EXERCISES [A]

Solve the following equations

1. $3x - 5 = x + 7$
2. $3y + 5 = y - 1$
3. $x - 6 = 5x + 14$
4. $3c + 3 = 10c - 4$
5. $3 + 3y = 7 + 5y$
6. $5r = 10r - 15$
7. $2x + 4 = 6x + 12$
8. $2(y + 4) = 3y + 8$
9. $0.3x + 2 = 14$
10. $0.5y - 3 = 0.2y$
11. $\frac{x-1}{2} = \frac{x+1}{3}$
12. $\frac{2y-1}{5} = \frac{y+1}{2}$
13. $0.52x - 3.9 = 0.26x$
14. $3.8x - 1.36 = 10.6x$
15. $4(x - 1) = 3(x + 2)$
16. $2 - y = 3y + 4y$
17. $20x = 10(5 - x)$
18. $9x + 1 = 2(1 + x)$
19. $(2x - 5) - 6(x + 1) = 0$
20. $\frac{1}{2}x = 10$
21. $5x - \frac{1}{4} = 3x + \frac{1}{2}$
22. $\frac{2}{3}x - 7 = \frac{3}{4}x + 2$
23. $\frac{n}{2} + \frac{n}{3} - \frac{n}{4} = 7$
24. $\frac{x-5}{2} = \frac{x-8}{3}$

Did you remember the proof

25. $\frac{x-4}{2} - \frac{5x+1}{3} + \frac{2x+1}{3} = \frac{x-2}{4}$
26. $\frac{3n-1}{4} - \frac{2n+1}{5} - \frac{3(n+1)}{4} + \frac{n+5}{3} = 0$
27. $6(2x + 3) - 3(x + 1) = 3(x + 2)$
28. $3(n + 1) + n^2 + 6 = n^2 + 12$
29. $0.96x - 0.64 + 0.04x = 11.46 - 1.72x + 0.14$
30. $\frac{2(3y-4)}{5} - 8(y + 1) = \frac{5-y}{4} + 35$

Literal Equations and Formulas [A]

A literal equation is an equation that contains letters other than the unknown. A formula is a rule written in the form of a literal equation. It tells how to find one quantity when the quantities on which it depends are given. Thus $A = \frac{1}{2}bh$ is a formula which tells how to find the area of a triangle when the base and altitude are known.

Example 1. Solve $3(x - a) = x + 3a$, for x

Solution
$$3x - 3a = x + 3a$$
$$3x - x = 3a + 3a$$
$$2x = 6a$$
$$x = 3a$$

PROOF Does $3(3a - a) = 3a + 3a$?
Does $9a - 3a = 6a$?
Does $6a = 6a$? Yes

Example 3 Solve $A = \frac{1}{2}bh$, for h

Solution M_2 $\quad 2A = bh$

D_b $\quad \frac{2A}{b} = h$

or $\quad h = \frac{2A}{b}$

PROOF Does $A = \frac{1}{2}b\left(\frac{2A}{b}\right)$?
Does $A = A$? Yes

[A] **EXERCISES**

Solve for x and y

1. $bx = 2bc$
2. $4x = -8m$
3. $ax = b$
4. $x + a = b$
5. $ax = ab$
6. $b^2x = 3b$
7. $mx = am^2$
8. $5x - a = b$
9. $x + a = 3a$
10. $ax + b = b$
11. $rxt = c$
12. $y - c = 2$
13. $x + 1 = a$
14. $\frac{1}{2}x = p$
15. $\frac{1}{3}y = c$
16. $ax - b = c$
17. $ax + a = 7a$
18. $3(m - x) = x - 7m$
19. $rx + s = t$
20. $0.05x = 2h$
21. $0.07y = 2k$
22. $cy = c^{n+1}$
23. $a^nx = a^{3n}$
24. $m^2x = m + 1$

Solve for the letter indicated

25. $A = \frac{1}{2}bh$ for b
26. $pv = c$ for p
27. $C = 2\pi r$ for r
28. $s = vt$ for v
29. $A = \frac{\pi r^2 E}{180}$ for E
30. $i = prt$ for r
31. $W = I^2R$ for R
32. $V = \frac{1}{3}bh$ for h
33. $S = 2\pi rh$ for r
34. $S = 180(n - 2)$ for n
35. $S = \frac{n}{2}(a + l)$ for a
36. $A = p(1 + rt)$ for t
37. $l = a + (n - 1)d$ for n
38. $A = \frac{1}{2}h(a + b)$ for b
39. $S = 2\pi r(r + h)$ for h
40. $s = \frac{1}{2}gt^2$ for g

Evaluation of Formulas [A]

Evaluating a formula consists in finding the value of one letter of the formula when the other letters are known If we wish to find the value of one letter of a formula for several different sets of values of the other letters it saves time and is easier if we first solve the formula for the unknown letter If there is only one evaluation to be made, it is usually easier to substitute directly in the formula and then solve for the required letter

Example Find the number of sides of the polygons the sum of whose interior angles are 720°, 1080°, 1260°, and 1620° respectively

Solution The formula is $S = 180(n-2)$, where $S =$ the sum of the number of degrees in the interior angles of a polygon and $n =$ the number of sides Our computation is shortened if we first solve the formula for n

$S = 180(n-2)$	Now substitute in $n = \frac{S}{180} + 2$
$S = 180n - 360$	When $S = 720$ $n = \frac{720}{180} + 2 = 6$
$180n = S + 360$	When $S = 1080$, $n = \frac{1080}{180} + 2 = 8$
$n = \frac{S + 360}{180}$	When $S = 1260$, $n = \frac{1260}{180} + 2 = 9$
$n = \frac{S}{180} + 2$	When $S = 1620$, $n = \frac{1620}{180} + 2 = 11$

EXERCISES

[A]

Make the evaluations necessary to answer each of the following questions Use $\pi = 3\,14$

1 The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$ What is the volume of a sphere whose radius is 2 feet?

2 How many gallons will a cylindrical pail contain if its height is 10 inches and the radius of its base is 5 inches? The required formula is $V = \pi r^2 h$ (1 gallon = 231 cubic inches)

3 Ohm's law is expressed by the formula $I = \frac{E}{R}$, where I is the number of amperes in the current, R the resistance in ohms, and E the potential difference in volts How much current will flow through a light bulb which has a resistance of 220 ohms and is connected to an electric-light outlet across which 110 volts of potential difference is maintained?

4 Find the voltage necessary to permit a current of 900 amperes to flow through a wire having a resistance of 11 ohms

5 In the formula $W = I^2R$ W is the number of watts I the current in amperes and R the resistance in ohms Find the resistance of a 60 watt lamp if it requires a $\frac{1}{2}$ ampere current

6 In the formula for the law of falling bodies $s = \frac{1}{2} gt^2$ s equals the distance the body falls in feet t the time the body falls in seconds and g the force of gravity equals about 32 2 A stone dropped from the top of a bridge struck the water in 3 seconds How high was the top of the bridge from the water?

7 Some hydrogen gas had a volume of 1843 cubic inches when under a pressure of 32 pounds What would be its volume when under a pressure of 100 pounds?

Use Boyle s law $\frac{v_1}{v_2} = \frac{p_2}{p_1}$

8 Find E in the formula $C = \dfrac{E}{R_1 + R_2}$ if $C = 22$ $R_1 = 4$ and $R_2 = 6$

9 Use the formula $C = 2\pi r$ to find the radius of a circle whose circumference is 25 inches

10 The formula for the area of a trapezoid is $A = \frac{1}{2} h(a + b)$ where h is the height and a and b the bases If each of three trapezoids has its lower base 12 inches and its upper base 8 inches what must be the heights of the trapezoids if their areas are 30 square inches 45 square inches and 70 square inches respectively?

Algebraic Representation [A]

In order to solve problems algebraically it is necessary to express number relations by the use of symbols The following exercises will give you practice in expressing such relationships

[A] **EXERCISES**

Express in symbols

1 One part of 10 is x What is the other part?

2 The sum of two numbers is 12 and one of them is a What is the other?

3 The difference between two numbers is a and the smaller number is b What is the larger number?

4 By how much does x exceed y?

5 How many cents are there in d dollars?

6 If in a two-digit number the units' digit is x and the tens' digit is y, what expression represents the number?

7 How far will a car traveling x miles an hour go in $(y + 2)$ hours?

8 How long will it take a plane flying at the rate of $(x + 30)$ miles an hour to go y miles?

9 At what rate must an automobile travel to go a miles in k hours?

10 If x is an integer, what expression will represent the next higher integer?

11 If Tom is x years old and Joe is 5 years older than Tom, what will represent the age of each 10 years from now?

12 Write three consecutive odd integers if the smallest one is t

13 How much will k pounds of coffee cost at n cents a pound?

14 If a floor is $(x + 2)$ feet wide and y feet long, how many square feet of carpet are required to cover it?

15 Express in cents, a dollars, b half dollars, and c cents

Verbal Problems [A]

When we can determine sufficient information of the right kind about an unknown number, we can relate the unknown to known numbers by means of an equation In the verbal problems of this chapter the necessary information has been assembled for you Your task is to organize this information into equation form, and then to solve the equation you obtain

Whether you consider yourself "good" or "poor" at problem solving, you can become an expert if you will train yourself to think systematically Thinking systematically means following a pattern of procedure A simple pattern for you to follow in problem solving is outlined on the next page

Steps in Solving Verbal Problems

1. Determine the question asked or the direction given.
2. Answer the question or follow the direction
   by using symbols to represent the unknowns to be found
   a. If only one number is to be found,
      let a letter represent it
   b. If more than one number is to be found,
      let a letter represent one of them,
      and then represent each of the others in terms of that one
3. Study the statement of the problem
   to find the relationships between the known and unknown numbers,
   and organize the information into equation form
4. Solve the equation.
5. Prove that the numbers found
   satisfy the information given in the problem

From time to time, on the following pages, special suggestions will be given to aid you in forming equations for the problems you are asked to solve

By studying two examples, let us see how the general steps outlined above are used in problem solving

Example 1 If 2 is added to 7 times a number, the result is 23 What is the number?

Solution

WHAT WE THINK	WHAT WE WRITE
1 The question is, "What is the number?"	
2 We answer the question ----→	Let $x =$ the number
3 We study the information and form the equation ---------→	$2 + 7x = 23$
4 We solve the equation --------→	$7x = 21$ $x = 3$
5 We prove that the value found satisfies the problem	PROOF When 2 is added to 7 times the number 3, the result is $2 + 21$, or 23 Since the problem states that the sum should be 23, we know that the number is 3

Example 2 Separate 80 into two parts such that the sum of one sixth of the smaller and one seventh of the larger shall be 12

Solution

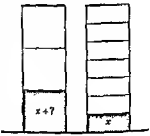
WHAT WE THINK	WHAT WE WRITE
1 The direction is Divide 80 into two parts	• •
2 We follow the direction	Let s = the small part and $80 - s$ = the large part
3 We study the information and form the equation	$\frac{1}{6}s + \frac{1}{7}(80 - s) = 12$
4 We solve the equation	$7s + 6(80 - s) = 504$ $7s + 480 - 6s = 504$ $s + 480 = 504$ $s = 24$ small number $80 - s = 56$ large number
5 We prove that the value found satisfies the conditions set forth in the problem	PROOF $24 + 56 = 80$ The sum of $\frac{1}{6}$ of 24 and $\frac{1}{7}$ of 56 is 12 Hence we know that the two parts are 24 and 56

In proving that a solution obtained for a verbal problem is correct we must do more than show that it satisfies the equation formed If we form a wrong equation its solution does not (except in a few rare instances) satisfy the conditions set forth in the problem We show that our solutions for verbal problems are correct by showing that they meet the conditions of the problem

EXERCISES

[A]

1 The difference between two numbers is 16 Three times the larger number is equal to seven times the smaller number What are the numbers?



2 Find two numbers whose sum is 58 and whose difference is 16

3 Two boys have a total of \$7 60 One boy has \$1 10 more than the other How much has each boy?

4 Separate 135 into three parts such that the second part is twice the first part and the third part is 10 more than the second

5 Alice and Jane collect plates as a hobby When the girls counted this morning, Jane had three times as many as Alice She then gave Alice 5 of hers Now Jane has only twice as many as Alice How many had each this morning?

Geometry Problems [A]

In solving geometry problems you should keep in mind these facts

1 The sum of the angles of a triangle is 180°
2 If two angles are complementary, their sum is 90°
3 If two angles are supplementary, their sum is 180°

Example One acute angle of a right triangle contains five times as many degrees as the other How many degrees are there in each acute angle?

Solution Let x = the number of degrees in the smaller acute angle

Then $5x$ = the number of degrees in the larger acute angle

From the fact that the acute angles of a right triangle are complementary, we form the equation

$$x + 5x = 90$$
$$6x = 90$$
$$x = 15, \text{ the number of degrees in the smaller acute angle}$$
$$5x = 75, \text{ the number of degrees in the larger acute angle}$$

PROOF Does $75 = 5(15)$ and does $15 + 75 = 90$? Yes

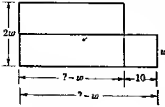
[A] **EXERCISES**

1. Two angles are complementary and their difference is 8 degrees Find the angles

2. Two angles are supplementary and one angle exceeds 3 times the other by 8 degrees Find the angles.

12 56
50

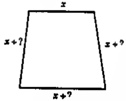
3 One angle of a triangle exceeds the second angle by 14° and exceeds the third angle by 22° Find the three angles of the triangle
10-40



4 The perimeter of a rectangle is 100 feet If the width is doubled and the length is decreased by 10 feet, the perimeter remains the same What are the width and length of the rectangle?
60

5 The difference between three times an angle and five times its complement is 30° Find the angle
33 1/2 22

6 The perimeter of a triangle is 80 feet The sum of the two longest sides is 58 and the difference of the two shortest sides is 3 Find the length of each side
10 14 14 18



7 The perimeter of an isosceles trapezoid is 56 Each of the equal sides is 4 more than the shortest side and 4 less than the longest side Find the length of each side of the trapezoid

Age Problems [A]

Problems called age problems are those in which the ages of the same persons are compared at different times in their lives The solution of such problems is easier if we keep in mind three simple facts

1 We all grow older together Each year each person s age increases one year

2 If one man s age in years is three times as great as another man s age in years today his age one year from today will *not* be three times the other man s age then

3 The difference in the ages of two persons is always the same throughout life

Example A man is 4 times as old as his son Sixteen years from now he will be only twice as old as his son How old is each at present?

Solution In order to form the correct equation we need to represent the father s age now and in 16 years and the son s age now and in 16 years We can represent this relationship by the following diagram

	AGE NOW	AGE IN 16 YEARS
Father	$4x$	$4x+16$
Son	x	$x+16$

Then from the relation, "Sixteen years from now he will be only twice as old as his son," we form the equation

$$4x + 16 = 2(x + 16)$$

Solving

$$4x + 16 = 2x + 32$$
$$2x = 16$$
$$x = 8$$
$$4x = 32$$

At present the son is 8 years old and the man is 32 years old

PROOF

Does $32 = 4 \times 8$? Yes

Does $32 + 16 = 2(8 + 16)$?
Does $48 = 2(24)$? Yes

EXERCISES

[A]

1. A man is three times as old as his son. Eleven years from now he will be only twice as old as his son. How old is each at present?

2. A father is twice as old as his daughter. In four years the father's age will be three times what the daughter's age was six years ago. How old is each at present?

3. The combined age of two boys is 25 years. Three years ago the age of one boy exceeded twice the age of the other boy by 1 year. Find the age of each boy.

4. The combined age of a mother and her two sons is 50 years. Four years ago the mother was 8 times as old as her older son. Four years from now she will be 4 times as old as her younger son. Find the age of each at present.

5. The combined age of three brothers is 45 years. Six years ago the oldest brother was twice as old as the youngest brother. Six years from now the combined ages of the two younger brothers will exceed their older brother's age by 15 years. How old is each of the brothers?

Motion Problems

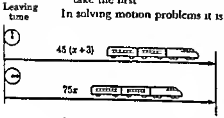
The relation between distance, time, and rate is given by the formula $d = rt$, where r is a constant rate. This relation may also be

written $r = \frac{d}{t}$ or $t = \frac{d}{r}$ If the distance is expressed in miles and the time in hours then the rate must be expressed in miles per hour (m p h)

Example A train leaves a station and travels at 45 miles an hour Three hours later a second train leaves and travels at 75 miles an hour How long will it take the second train to overtake the first?

Solution Let $x =$ the time in hours for the second train to overtake the first

In solving motion problems it is best to tabulate the data



	r	$\times$ t	$=$ d
First train	45	$x+3$	$45(x+3)$
Second train	75	x	$75x$

Since the trains travel the same distance, the equation is

$$75x = 45(x+3)$$

Solving

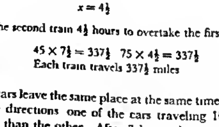
$$75x = 45x + 135$$
$$30x = 135$$
$$x = 4\tfrac{1}{2}$$

It takes the second train $4\frac{1}{2}$ hours to overtake the first

PROOF $45 \times 7\frac{1}{2} = 337\frac{1}{2}$ $75 \times 4\frac{1}{2} = 337\frac{1}{2}$
Each train travels $337\frac{1}{2}$ miles

EXERCISES

1 Two cars leave the same place at the same time and travel in opposite directions one of the cars traveling 10 miles an hour faster than the other After 3 hours they are 240 miles apart What is the rate of each car in miles per hour?



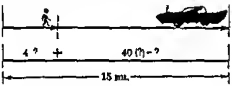
2 Two planes which are 1060 miles apart leave at the same time and fly toward each other, meeting in 4 hours If their rates differ by 15 miles an hour, what is the rate of each plane?

3 One car traveling at the rate of 60 miles an hour is 4 miles behind another car traveling at 50 miles an hour How many minutes will it take the faster car to overtake the slower?

4 A man travels from one city to another at the rate of 40 miles an hour He returns at the rate of 60 miles an hour If the trip takes 8 hours, how far apart are the cities?

5 Two men travel toward each other from points 540 miles apart If their rates are 48 and 60 miles an hour respectively when will they meet if they started at the same time?

6 A boy starts to walk to a town 15 miles away at the rate of 4 miles an hour After walking for a time he is taken to the town by a friend traveling by car at the rate of 40 miles an hour If the boy s total time reaching the town is $1\frac{1}{2}$ hours, how far does he ride? 10 miles



7 A plane has a cruising speed of 280 miles an hour and the wind velocity is 40 miles an hour

a What is the rate of the plane with the wind?

b What is the rate against the wind?

c How far can the plane fly against the wind and return in 7 hours?

8 A plane can fly a certain distance in 6 hours with the wind but can return only three fourths the distance in the same time If the speed of the plane in still air is 200 miles an hour, find the velocity of the wind

9 A man drove at the rate of 50 miles an hour while out side a city and 30 miles an hour while within the city If a trip of 30 miles took him 40 minutes, how much of the trip was outside the city?

Lever Problems [4]

A lever is a bar which can rotate on a point of support called the fulcrum. In the figure below, point F indicates the fulcrum, w_1 a weight at the distance d_1 from F, and w_2 a weight at the distance d_2 from F. Neglecting friction and the weight of the lever, a lever will balance when

$w_1d_1 = w_2d_2$

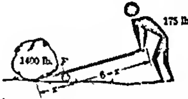
Example. A bar 6 feet long is to be used as a lever. Where would a man weighing 175 pounds have to place the fulcrum in order to lift an object weighing 1400 pounds?

Solution

Let x = the length of the lever arm from the object to the fulcrum.
Then $6 - x$ = the length of the lever arm from the fulcrum to the man.

The two weights will be in equilibrium when

$$1400\,x = 175(6 - x)$$
$$1400\,x = 1050 - 175\,x$$
$$1575\,x = 1050$$
$$x = \tfrac{2}{3}$$



Therefore, when the fulcrum is placed at a distance of $\frac{2}{3}$ of a foot from the object, the weights are in equilibrium. In order to lift the object the man would have to place the fulcrum less than $\frac{2}{3}$ of a foot from the object.

PROOF. Left to student.

EXERCISES

[A]

1. Where will a 12-foot beam balance if a 125-pound weight is placed on one end and a 75-pound weight is placed on the other end?

2. If a 240-pound weight is to rest on a beam 12 feet from the fulcrum, what weight must be placed on the beam on the other side of the fulcrum and 8 feet from it so that the beam will be in equilibrium?

3. Two boys weighing 60 and 75 pounds respectively sit on opposite ends of a seesaw 12 feet long. Where must the fulcrum be placed in order that they may balance each other?

4. Rosemary and Constance together weigh 135 pounds The girls balance on a seesaw when Rosemary is 6 feet from the fulcrum and Constance is 9 feet from it How much does each girl weigh?

5. How heavy an oil drum can a man, by exerting a force of 200 pounds, lift with a crowbar 5 feet in length if the fulcrum is 6 inches from the drum?

6. Two boys balance on a seesaw when they are respectively 5 feet and 6 feet from the fulcrum The larger boy takes his little sister, who weighs 30 pounds, on his lap and has to move 1 foot nearer the fulcrum in order to balance the other boy How much does each boy weigh?

Mixture Problems

In solving mixture problems, you will find it helpful to use sketches and to imagine that you are actually blending the ingredients, as suggested by the following examples

Example 1 How many pounds of cashew nuts worth 75 cents a pound must be mixed with pecans worth \$1 50 a pound to obtain 50 pounds of mixed nuts worth \$1 00 a pound?

Solution Let $x =$ the number of pounds of 75-cent nuts
Then $50 - x =$ the number of pounds of \$1 50 nuts

Since	the value of the x pounds of 75-cent nuts	+	the value of the $(50 - x)$ pounds of \$1 50 nuts	=	the value of 50 pounds of the \$1 00 mixture
then	$75x$	+	$1\ 50(50 - x)$	=	$1\ 00(50)$

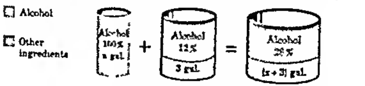
M_{100}, $75x + 150(50 - x) = 100(50)$

$$75x + 7500 - 150x = 5000$$
$$-75x = -2500$$
$$x = 33\tfrac{1}{3}, \text{ pounds of cashews}$$
$$50 - x = 16\tfrac{2}{3}, \text{ pounds of pecans}$$

PROOF Left to student

Example 2 How much pure alcohol must be added to 3 gallons of a 12% solution to make a 28% solution?

Solution Let x = the number of gallons of pure alcohol required



Since	x gallons of pure alcohol	+	alcohol in 3 gallons of a 12% solution	=	alcohol in $(x+3)$ gallons of a 28% solution
then	$1.00\,x$	+	$12(3)$	=	$.28(x+3)$
M_{100}			$100\,x + 12(3)$	=	$28(x+3)$
			$100\,x + 36$	=	$28\,x + 84$
			$72\,x$	=	48
			x	=	$\frac{2}{3}$ gallons of pure alcohol to be added

PROOF Left to student

EXERCISES

A

1. The doctor told Nurse Barton to dilute the antiseptic solution for the Jones baby from 25% to 15%. How much water must Miss Barton add to 1 pint of the 25% solution to get a 15% solution?

2. A garageman has 500 gallons of alcohol that is 75% pure. He wishes to add water until it is 40% pure. How much water must be added?

3. A grocer blends two grades of tea which are worth \$1.20 and \$1.60 a pound respectively. How much of each kind of tea must he use to make a mixture of 50 pounds which he can sell at \$1.50 a pound?

4. How much cream which is 25% butterfat should be added to 1200 pounds of milk which is 3% butterfat to produce a milk testing 4% butterfat?

5. One alloy is 25% silver and another is 40% silver. How much of each should be used to produce 60 pounds of an alloy that is 30% silver?

6 A confectioner wishes to mix chocolates worth 80 cents a pound with 40 pounds of bonbons worth 65 cents a pound to make a mixture worth 75 cents a pound How many pounds of chocolates should he use?

7. How much alcohol must be added to a mixture of 12 ounces of alcohol and 30 ounces of water to produce a mixture that is 60% alcohol?

8 An automobile radiator holds 4 gallons and contains 25% antifreeze How much of the solution must be drained from the radiator and antifreeze added to make the solution test 35% antifreeze?

9. How many tons of 8% copper ore and of 3% ore should be mixed to make 400 tons of 6% ore?

Checking Your Understanding of Chapter 2

If you expect to understand algebra and to use it easily, do not move on to the next chapter until you are sure that you understand this chapter

Make sure that you know

PAGE

1 What an equation is This includes knowing the difference between conditional and identical equations 39

2 What it means to solve an equation 40

3 What a root of an equation is 40

4 That a value found for an unknown in an equation cannot be accepted as a root of the equation without proof 42

5 What is meant by the expression *first-degree equation in one unknown* 40

Make sure that you can easily

6 Carry out the steps in solving a first-degree equation in one unknown This includes literal equations 42

7 Carry out the steps in proving that the value found for the unknown in an equation is a root of the equation 42

8 Evaluate a formula 46

9 Analyze a verbal problem and translate it into equation form 48

Make sure that you understand the meaning of, and can spell the words in, the following expressions

MATHEMATICAL VOCABULARY

	PAGE		PAGE
assumption	41	member of an equation	39
conditional equation	39	root of an equation	40
formula	44	satisfy	40
identity	40	solution	40

CHAPTER REVIEW

[A]

Solve

1. $5x - 8 = x + 12$
2. $2(y - 5) = 3y - 2$
3. $0.3x + 2 = 17$
4. $(2x - 3) - 3(x - 6) = 4$
5. $\frac{2}{3}y - 5 = \frac{3}{4}y + 4$
6. $\frac{n-2}{4} - \frac{2n+3}{5} + \frac{5n}{2} = 13$

Solve

7. $ax + bx = 3ax - c$ for x
8. $b^2x = 4b$ for x
9. $A = P(1 + rt)$ for t
10. $A = \pi r^2 + 2\pi rh$ for h

Evaluate

11. $V = e^3$ when $e = 5$
12. $l = a + (n - 1)d$ when $a = 3$, $n = 5$, $d = 4$

CHAPTER TEST

[A]

Part I

1. Give an example of a. An identity b. A conditional equation

Solve:

2. $4x - 7 = x + 11$
3. $2x + 4 = 14 - 3x$
4. $0.52y - 0.40 = 10.00$
5. $\frac{3}{4}y = 12$
6. $r^2 + 5 = r^2 + 10r - 15$
7. $2(x - 5) - 6(x + 3) = 0$
8. $\frac{1}{2}(x + 6) = \frac{1}{3}(x - 8)$
9. $2x(3x + 4) + 9 = 3x(2x + 1) - 1$
10. $\frac{x-4}{2} - \frac{5x+1}{3} = \frac{2x-3}{4} + \frac{7}{4}$

Part II

Solve for x

1 $ax + b = c$

2. $sx = s^3$

3 $3(a - x) = 4(x - 2a) - 10a$

Solve

4 $V = \frac{1}{3}bh$ for b 5 $A = P(1 + rt)$ for P 6 $rsx = 0$ for x

7 $S = \frac{n}{2}(a + 1)$ for n 8. $F = \frac{9}{5}C + 32$ for C

9 Using the formula $S = 4\pi r^2$, find the area of a sphere whose radius is 4 inches ($\pi = 3.14$)

10 Find n in the formula $S = 180(n - 2)$ when $S = 1440$

Part III

1 How many gallons of pure alcohol must be added to 4 gallons of a 10% solution to make a 20% solution?

2. How much orange juice at 90 cents a gallon should be mixed with grapefruit juice worth 60 cents a gallon to make 100 gallons of a mixture worth 80 cents a gallon?

3. A motorist drove to a certain city in 3 hours, and returned by another route which was 22 miles longer On the return trip he traveled 10 miles an hour faster and it took him 2 hours and 50 minutes Find the length of the shorter route

4 Weights of 60 and 90 pounds are placed at the ends of a lever 9 feet long Where should the fulcrum be placed to make the lever balance?

5 Sam is twice as old as Joan, and next year the sum of their ages will be 5 times as much as Joan's age was two years ago How old are Sam and Joan?

Mathematics in Engineering

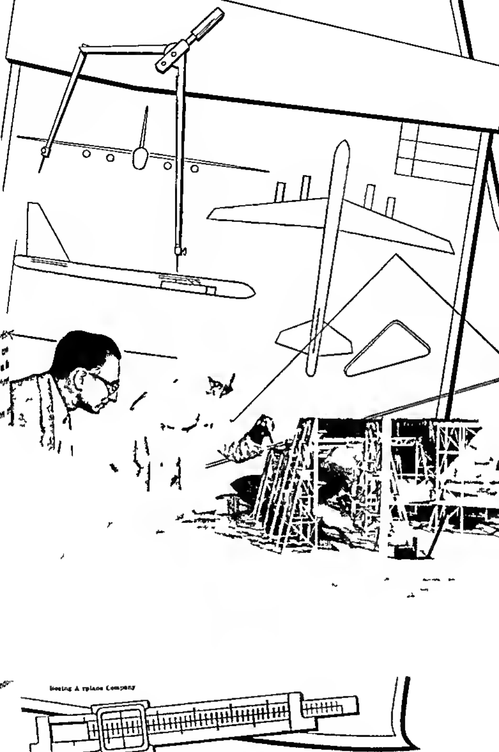
In recent years there has been an increasing demand for scientists, engineers, technicians, and craftsmen. Much of this demand has come from industry and has been caused by the rapid expansion in the manufacture of mechanical appliances and by the manufacturers' increasing interest in improving their methods of production.

There are many different kinds of engineers. Among them are electrical engineers, mechanical engineers, civil engineers, chemical engineers, and engineers with many other kinds of specialized knowledge, such as the aeronautical engineers in the picture.

The following high school subjects are useful for future engineers:

English	Trigonometry	Foreign languages
Algebra	Physics	Industrial arts
Plane geometry	Chemistry	Salesmanship
Solid geometry	Mechanical drawing	Typewriting
	Social studies	

Most engineers are college graduates, and a considerable number of them have had from one to three years of graduate work, specializing in their particular field of interest. During the first two years in an engineering school the student usually studies college algebra, trigonometry, analytic geometry, and calculus, but some colleges require the student to have advanced algebra and trigonometry before entering. All these courses are a necessary foundation for advanced courses in mathematics.



Boeing Airplane Company

CHAPTER 3

Type Products, Factoring, Equations

In this chapter you multiply, and you analyze multiplication

In your first course in algebra you learned short methods of finding products such as $(x+3y)(5x-2y)$, $(a-6)(a+6)$, and $(m+2)(m+2)$, and could state the products without the use of pencil and paper. Products which can be found by short cuts are called special products. The products above are of three types, the second and third products being special types of the first. In the first part of this chapter you will study special products. Then you will make use of special products in factoring. In the latter part of the chapter you will use special products and factoring in solving equations.

[A]

REVIEW EXERCISES

Find the following products

1. $8(a-b+c)$
2. $x^2(x^2-x+3)$
3. $3abc(a+b-c)$
4. $5xy(x^2+y^2)$
5. $3a^n(a^n+a^{n-1})$
6. $-4x^3(x^2+x-1)$
7. $-5y^3(y^3-3y^2-y)$
8. $ax^2(1-2x^2-3x^4)$
9. $6xy(\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{6}y^2)$
10. $5x^a(x^{2a}-3x^a+x^{a+1})$

Product of Two Binomials [A]

If two binomials have no like terms, their product is a polynomial of four terms.

For example, $(a+b)(c+d) = ac+bc+ad+bd$

If two binomials have like terms, their product is either a binomial or a trinomial. The product at the right is a trinomial.

$$\begin{array}{r} 2x+3y \\ \underline{x-4y} \\ 2x^2+3xy \\ \underline{-8xy-12y^2} \\ 2x^2-5xy-12y^2 \end{array}$$

Let us see how this product is found mentally. The $2x^2$ is found by multiplying $2x$ by x, the first terms of the binomials, the $-12y^2$ is found by multiplying $3y$ by $-4y$, the last terms of the binomials, and $-5xy$ is found by adding $3xy$ and $-8xy$, which are the cross products of the multiplication.

Example 1 Find the product $(3a+b)(2a-5b)$

Solution The first term of the product is $(3a)(2a) = 6a^2$, the last term of the product is $(b)(-5b) = -5b^2$, one cross product is $2ab$, the other is $-15ab$, their sum is $-13ab$

Then

$(3a+b)(2a-5b) = 6a^2-13ab-5b^2$

$$\begin{array}{c} 2ab \\ (3a+b)(2a-5b) \\ -15ab \end{array}$$

The product of two binomials having the form $(ax + by)(cx + dy)$
equals the product of the two first terms,
plus the algebraic sum of the cross products,
plus the product of the two last terms.

Example 2 Find the product $(2x - y)(2x + y)$

Solution. The product of the two first terms is $4x^2$, the cross products are $-2xy$ and $+2xy$ and their sum is 0, the product of the two last terms is $-y^2$

Then $$(2x - y)(2x + y) = 4x^2 - y^2$$

The product is a binomial, since the sum of the cross products is zero

Example 3 $(5c - d)^2 = (5c - d)(5c - d) = ?$

Solution The two factors are identical $(5c)(5c) = 25c^2$, $(5c)(-d) + (-d)(5c) = -10cd$, and $(-d)(-d) = d^2$ Then
$$(5c - d)(5c - d) = 25c^2 - 10cd + d^2$$

Do you see that Examples 2 and 3 are special cases of the general type $(ax + by)(cx + dy)$?

EXERCISES

[A]

Study the examples above and find the following products mentally Check the first five products, using the long method of multiplication

1. $(a + 3)(a + 2)$
2. $(a - 3)(a - 2)$
3. $(x + 4)(x - 3)$
4. $(c - 5)(c - 2)$
5. $(1 - 7x)(1 + 9x)$
6. $(x + 3)(x + 3)$
7. $(2x - 1)^2$
8. $(2x + 4)^2$
9. $(7c + 3d)(7c - 3d)$
10. $(3xy - 1)(4xy + 2)$
11. $(rs + 3)(rs - 1)$
12. $(2x - 9)(3x + 10)$
13. $(m^3 - n^2)(m^3 + n^2)$
14. $(c^3 - 8d)(c^3 + 8d)$
15. $(3x - 5y)(2x + y)$
16. $(x - y)(r + s)$
17. $(2a - 3)(3 + a)$
18. $(2x - 3y)(5x - y)$
19. $(x^2 - 1)(x^2 + 1)$
20. $(a^2 - 3)(a^2 + 3)$
21. $(a^2 - 5)$[illegible]
22. $(ab - 9)(ab + 10)$
23. $(m^3n + 8)(m^3n - 6)$
24. $(3 - c^2d)(5 - 4c^2d)$

25 $(7x - y^2)(3x + 2y^2)$
26 $(x - \frac{1}{2})(x + \frac{1}{2})$
27 $(a - \frac{1}{4})(a - \frac{3}{4})$
28 $(c - \frac{1}{2}d)(c - \frac{1}{4}d)$
29 $(a + 1)(a + 3)$
30 $(b - 2)(b - 3)$
31 $(x + 5)^2$
32 $(y - 1)^2$
33 $(2ax - bc)(3ax + bc)$
34 $(ab - cd)^2$
35 $(a^n + 1)(a^n + 2)$
36 $(x^n + 2)(x^n + 3)$
37 $(a^{2n} - 1)(a^{2n} + 3)$
38 $(x^n - y^{3n})(x^n + 2y^{3n})$
39 $(B^a + A^b)(B^a + A^b)$
40 $(x^{n+1} - 2)(x^{n+1} + 5)$

Squaring a Binomial [A]

If we square $x + y$, the result is $x^2 + 2xy + y^2$, and if we square $x - y$, we get $x^2 - 2xy + y^2$. These two products are special types of the form

$$(ax + by)(cx + dy)$$

$$\begin{array}{r} x + y \\ \underline{x + y} \\ x^2 + xy \\ \underline{xy + y^2} \\ x^2 + 2xy + y^2 \end{array} \qquad \begin{array}{r} x - y \\ \underline{x - y} \\ x^2 - xy \\ \underline{-xy + y^2} \\ x^2 - 2xy + y^2 \end{array}$$

The square of the sum (or difference) of two numbers is equal to the square of the first number plus (minus) twice the product of the two numbers, plus the square of the second number

EXERCISES [A]

Find the squares

1 $(x + 3)^2$
2 $(x - 4)^2$
3 $(a + 5)^2$
4 $(c - 6)^2$ c - 6
5 $(h + 7)^2$
6 $(y - 8)^2$
7 $(h - x)^2$
8 $(1 - y)^2$
9 $(2 + m)^2$
10 $(x - 3y)^2$
11 $(ab + 1)^2$
12 $(cd - 9)^2$
13 $(m^2 + 5)^2$
14 $(n^2 + 10)^2$
15 $(x - ab)^2$
16 $(x - \frac{1}{2})^2$
17 $(y + \frac{1}{3})^2$
18 $(2x - \frac{1}{2})^2$
19 $(3c - 4d)^2$
20 $(1x + 5y)^2$
21 $(x^a + y^b)^2$
22 $(m^a - y^b)^2$
23 $(3a^x - 4b^y)^2$
24 $(10^a - 10^b)^2$
25 $(10^x + 5^y)^2$
26 $(4m^3 - \frac{1}{2}y)^2$
27 $(2a^x - \frac{1}{2}b^x)^2$

Product of the Sum and Difference of Two Numbers [A]

When the sum of two numbers is multiplied by their difference, the sum of the cross products is zero, making the product a binomial

$$\begin{array}{r} a+b \\ \underline{a-b} \\ a^2+ab \qquad \\ \underline{\qquad -ab-b^2} \\ a^2 \qquad\quad -b^2 \end{array}$$

> The product of the sum and difference of two numbers is equal to the difference of their squares.

Example Find by inspection $(x-6y)(x+6y)$

Solution The square of x is x^2 and the square of $6y$ is $36y^2$

Then $(x-6y)(x+6y)=x^2-36y^2$

[A]

Find the products

1 $(x+1)(x-1)$	14 $(xy-7)(xy+7)$
2 $(x-2)(x+2)$	15 $(1-c^2)(1+c^2)$
3 $(y+5)(y-5)$	16 $(m-np)(m+np)$
4 $(c-8)(c+8)$	17 $(2-rs)(2+rs)$
5 $(5a-x)(5a+x)$	18 $(\frac{1}{2}-p)(\frac{1}{2}+p)$
6 $(x-\frac{1}{3})(x+\frac{1}{3})$	19 $(x+5)(x-5)$
7 $(c+2d)(c-2d)$	20 $(ab-3)(ab+3)$
8 $(m-3n)(m+3n)$	21 $(x^n-4)(x^n+4)$
9 $(c^2+6)(c^2-6)$	22 $(y^a-3)(y^a+3)$
10 $(x^3-7)(x^3+7)$	23 $(x^a-y^b)(x^a+y^b)$
11 $(x-\frac{1}{3})(x+\frac{1}{3})$	24 $(y^a+h^n)(y^a-h^n)$
12 $(3c+\frac{2}{3})(3c-\frac{2}{3})$	25 $(4x^a-5y^b)(4x^a+5y^b)$
13 $(ab+1)(ab-1)$	26 $(a^mb^n+1)(a^mb^n-1)$

Binomials Used as Monomials [A]

Sometimes a seemingly difficult multiplication can be done mentally

Example 1 $[(c+d)+1][(c+d)-1]=?$

Solution Let $x=c+d$

Then $[(c+d)+1][(c+d)-1]=[x+1][x-1]=x^2-1$

Replacing x by $c+d$ $\quad x^2-1=(c+d)^2-1=c^2+2cd+d^2-1$

Then $[(c+d)+1][(c+d)-1]=c^2+2cd+d^2-1$

Example 2 Find by inspection $[(a+b)-3][(a+b)+5]$

Solution. This example is like the exercises on page 66, except that it contains the binomial $a+b$ instead of a monomial

Let $a+b=x$

Then $[(a+b)-3][(a+b)+5]=[x-3][x+5]=x^2+2x-15$

Replacing x by $a+b$,

$$x^2+2x-15=(a+b)^2+2(a+b)-15$$

Then $[(a+b)-3][(a+b)+5]=a^2+2ab+b^2+2a+2b-15$

After you have had sufficient practice, you will be able to find products of types like Examples 1 and 2 without substituting a monomial for the binomial

EXERCISE [A]

Find by inspection the following products

1. $[(c+d)-5][(c+d)+5]$
2. $[(c-d)+3][(c-d)-3]$
3. $[(2a-b)-m][(2a-b)+m]$
4. $[(x+y)+2][(x+y)+3]$
5. $[(m-n)+4][(m-n)-2]$
6. $[c+(m+n)][(c-(m+n)]$
7. $[2m-(n+p)][2m+(n+p)]$
8. $[4-(3c-1)][4-(3c-1)]$
9. $[1-(r-s)][1+2(r-s)]$
10. $[5-(x+y)][4-(x+y)]$

Example 3. Multiply $(x+y-c)(x+y+c)$

Solution Let $x+y=m$

Then $[x+y-c][x+y+c]=[m-c][m+c]=m^2-c^2$

Replacing m by $x+y$,

$$m^2-c^2=(x+y)^2-c^2=x^2+2xy+y^2-c^2$$

Then $[x+y-c][x+y+c]=x^2+2xy+y^2-c^2$

EXERCISES [A]

Find the products

1. $(x+y+3)(x+y+3)$
2. $(a+b-c)(a+b+c)$
3. $(x-y+7)(x-y+2)$
4. $(r+s-t)(r+s+t)$

5 $(3r+2s+7)(3r+2s-8)$	8 $(x-2y+3)(x-2y-6)$
6 $(x-2y-4)(x-2y+4)$	9 $(x^2-x-6)(x^2-x+6)$
7 $(m-n+p)(m-n-p)$	10 $(x^2+2x-1)(x^2+2x+5)$

Example 4 Multiply $x+y-t$ by $x-y+t$

Solution The signs of y and t in the multiplicand are opposite to the signs of y and t in the multiplier By using parentheses preceded by plus and minus signs the product can be written as the sum of two numbers times their difference

$$\begin{aligned}[x+y-t][x-y+t] &= [x+(y-t)][x-(y-t)] \\ &= x^2-(y-t)^2 \\ &= x^2-(y^2-2ty+t^2) \\ &= x^2-y^2+2ty-t^2\end{aligned}$$

[A]

Multiply by inspection

1 $(a+b-c)(a-b+c)$	4 $(h-k+4)(h+k-4)$
2 $(c-t+y)(c+t-y)$	5 $(m+3y-2z)(m+3y+2z)$
3 $(m-n-1)(m+n+1)$	6 $(m+2n+4)(m+2n+3)$

7 $(3x-4y-5z)(3x+4y+5z)$

8 $(3a+b+d)(3a-b-d)$	11 $(x^2+x+1)(x^2+x+1)$
9 $(x^2-xy+y^2)(x^2+xy+y^2)$	12 $(x^2-x+1)(x^2+x+1)$
10 $(c^2-cd+d^2)(c^2+cd+d^2)$	13 $(x^2-x+1)(x^2+x-1)$

Factoring [A]

Factoring a number is the process of finding two or more expressions whose product equals the number Thus the factors of 10 are 5 and 2 since the product of 5 and 2 is 10 Also, the factors of a^2-b^2 are $a+b$ and $a-b$ since $(a+b)(a-b)=a^2-b^2$

The Common Monomial Factor [A]

If the expression a^2-ab+b^2 is multiplied by $2a$, the product is $2a^3-2a^2b+2ab^2$ In this case we are finding the product when its factors are given Factoring is the reverse of this operation If we are asked to factor $2a^3-2a^2b+2ab^2$, we must find the monomial divisor $2a$ as one factor and then by division find the quotient a^2-ab+b^2 as the other factor

Example 1 Factor $7x^2 - 21x$

Solution The largest monomial factor (divisor) of $7x^2 - 21x$ is $7x$. The quotient found by dividing $7x^2 - 21x$ by $7x$ is $x - 3$. Then the factors of $7x^2 - 21x$ are $7x$ and $x - 3$. The solution is written

$$7x^2 - 21x = 7x(x - 3)$$

Example 2 Factor $6a^2b^2 - 12a^3b + 30a^2b^4$

Solution The greatest common divisor (G.C.D.) of 6, 12, and 30 is 6. The G.C.D. of a^2, a^3, and a^2 is a^2. The G.C.D. of b^2, b, and b^4 is b. Then the largest monomial factor is $6a^2b$.

Then $6a^2b^2 - 12a^3b + 30a^2b^4 = 6a^2b(b - 2a + 5ab^3)$

In factoring algebraic expressions we usually do not factor the monomial factor into its prime factors (a prime factor is one that cannot be factored). Notice that $6a^2b$ is not written $2 \cdot 3aab$.

[A] **EXERCISES**

Factor the following polynomials

1. $6x - 12$	11. $\pi r^2 + \pi rh$
2. $x^3 - x$	12. $\frac{1}{3}\pi r^2 + \frac{1}{3}\pi R^2 + \frac{1}{3}\pi rR$
3. $\pi R - \pi r$	13. $x^{m+1} - x^m$
4. $4a^3bx - 4bx$	14. $y^{2m} - y^m$
5. $a^2bc - ab^2$	15. $c^x - 3c^{3x}$
6. $5m^2 - 10m$	16. $h^2 - 4h^{n+2}$
7. $7cd - 14c^2d^2$	17. $2a^3 - 6a^2y$
8. $8a^3 - 16a^2$	18. $x^k - 6x^{k+2}$
9. $x^2y - xy^2$	19. $7ab^m - 21a^{c+1}b^c$
10. $15x^3 - 10x^2 + 5x$	20. $18m^x - 27x^{2m}$

Factoring Trinomials [A]

Some trinomials can be factored and some are prime (cannot be factored). We shall now factor trinomials which are the products of two binomials. If a trinomial of the form $ax^2 + bxy + cy^2$ can be factored into two binomial factors, we can factor it by the *guess* method. Study the two examples and their solutions.

Example 1 Factor $x^2 - 2x - 15$

Solution 1 If this trinomial can be factored into two binomials these first terms are x and x.

2 Since the sign of 15 is minus one of the signs of the second terms is plus and the other is minus

$$(x + \quad)(x - \quad)$$

3 Since the absolute value of the product of the second terms of the binomials is 15 these terms may be 5 and 3 or 15 and 1 Let us try 5 and 3 The sum of the cross products is $+2x$ It should be $-2x$ Then $x+5$ and $x-3$ are not the factors of $x^2 - 2x - 15$

$$(x+5)(x-3)$$

4 Let us try changing the signs of the second terms of the factors Now the sum of the cross products is $-2x$ Since $(x-5)(x+3) = x^2 - 2x - 15$ the factors of $x^2 - 2x - 15$ are $x-5$ and $x+3$ Then

$x^2 - 2x - 15 = (x-5)(x+3)$

$$(x-5)(x+3)$$

Example 2 Factor $10x^2 - 23xy + 12y^2$

Solution 1 We can factor $10x^2$ into $5x$ and $2x$ or $10x$ and x Let us try $10x$ and x as the first terms of the factors

$$(10x \quad)(x \quad)$$

2 Since the sign of $12y^2$ is plus the signs of the second terms of the factors are alike Since the sign of $23xy$ is minus the signs of the second terms of the factors are minus

$$(10x - \quad)(x - \quad)$$

3 For the second terms of the factors we may use $-6y$ and $-2y$ $-12y$ and $-y$ or $-4y$ and $-3y$ Let us first try $-3y$ and $-4y$ The sum of the cross products is $-43xy$ and not $-23xy$ So we must try another combination

$$(10x - 3y)(x - 4y)$$

4 Let us try $2x$ and $5x$ for the first terms of the factors and $-3y$ and $-4y$ for the second terms The sum of the cross products is $-23xy$ Then

$$(2x-3y)(5x-4y)$$

$$10x^2-23xy+12y^2=(2x-3y)(5x-4y)$$

If the product of $2x-3y$ and $5x-4y$ were not $10x^2-23xy+12y^2$, we should continue with other combinations until we had the correct one or until we had tried all of them and knew the polynomial was prime

EXERCISES

(A)

Factor the following

1. x^2+2x+1
2. $c^2+8c+16$
3. x^2-2x+1
4. $x^2-8x+16$
5. $m^2-2m-24$
6. $y^2-4y-32$
7. $c^2-12-4c$
   (Rearrange into descending powers of c)
8. $b^2+12-7b$ (b-3)(b-4)
9. $h^2-9h+20$
10. k^2-k-6
11. $3a^2+7a+4$
12. $x^2-3x-70$
13. $2x^2+7x+3$
14. $10x^2+17x+3$
15. y^2-6-5y
16. $a^2-10-3a$
17. x^2-1
   (The sum of the cross products is zero)
18. $4x^2-9$
19. c^2-64
20. y^2-y-20
21. $p^2-36+9p$
22. $1-6x+9x^2$
23. $1-10c+25c^2$
24. $c^2+2cd+d^2$
25. $c^2-2cd+d^2$
26. $x^2+2xy-15y^2$
27. $x^2-2xy-15y^2$
28. $m^2-mn-56n^2$
29. $3x^2-10x+8$
30. $8a^2+2a-21$
31. $2p^2-3p-5$
32. $3c^2+4c-7$
33. $13x^2-5xy-8y^2$
34. $3x^4-2x^2-5$
35. $a^{12}+4a^6+4$
36. x^8-12x^4+36
37. $2c^2-3cd-2d^2$
38. $3x^2+2xy-y^2$
39. $2a^2-5ab+2b^2$
40. $3-2a-8a^2$

41 $1 - 6a + 9a^2$

42 $5x^2y^2 - xy - 4$

43 $x^2y^2 - 7xy + 10$

44 $c^2d^2 - 10cd + 16$

45 $a^{2n} + 10a^n + 16$

46 $c^{2n} + 8c^n + 12$

47 $x^{2a} - x^a - 20$

48 $y^{2b} - y^b - 42$

49 $9a^{2x} + 12a^xb^y + 4b^{2y}$

50 $a^{4x} - 6a^{2x} + 9$

51 $3x^{2n} - 13x^ny + 4y^2$

52 $2x^{2a} + x^a - 3$

Perfect-Square Trinomials [A]

If the two binomial factors of a trinomial are alike the trinomial is the square of either of them Since $x^2 - 8x + 16 = (x - 4)(x - 4) = (x - 4)^2$ it is a perfect square trinomial A perfect square trinomial is easily recognized since it has two positive perfect square terms and has a third term whose absolute value is twice the product of the square roots of the perfect square terms

Example Is $y^2 + 4 + 4y$ a perfect square? If it is what is its square root?

Solution y^2 is the square of y and 4 is the square of 2 Two times y times 2 is $4y$ Then $y^2 + 4 + 4y$ is a perfect square $y^2 + 4y + 4 = (y + 2)^2$ and a square root is $y + 2$

EXERCISES

[A]

Which of the following trinomials are perfect squares? Factor the perfect squares

1 $m^2 - 2m + 1$

2 $y^2 + 6y + 9$

3 $x^2 + 5x + 25$

4 $y^2 - 10y + 25$

5 $h^2 + 4h + 8$

6 $4x^2 + 4x + 1$

7 $9a^2 - 12a + 16$

8 $25x^2 - 10x + 4$

9 $4c^2 - 20cd + 25d^2$

10 $4x^2 - 28x + 49$

Assuming that the radicands are positive numbers find the principal square roots

11 $\sqrt{x^2 - 8x + 16}$

12 $\sqrt{h^2 + 4h + 4}$

13 $\sqrt{9x^2 - 24x + 16}$

14 $\sqrt{36 + 36x + 9x^2}$

15 $\sqrt{25m^2 - 10m + 1}$

16 $\sqrt{x^2 + x + \frac{1}{4}}$

17 $\sqrt{4a^{2x} + 4a^x + 1}$

18 $\sqrt{c^2 - 5c + \frac{25}{4}}$

The Difference of Two Squares[1A]

If the sum of two numbers, a and b, is multiplied by their difference, the product is the difference of their squares, $a^2 - b^2$

$$\begin{array}{rr} a + b & \\ a - b & \\ \hline a^2 + ab & \\ - ab & - b^2 \\ \hline a^2 & - b^2 \end{array}$$

Then
$$a^2 - b^2 = (a + b)(a - b)$$

Since the first term of the product is the square of the first number, the first number is the square root of the first term of the product Likewise the second number is the square root of the second term of the product

> The difference of the squares of two numbers
> is equal to the product of
> the sum of the numbers and the difference of the numbers

Example 1 *Factor* $9x^2 - 25$

Solution. A square root of $9x^2$ is $3x$ and a square root of 25 is 5 One factor of $9x^2 - 25$ is $3x + 5$ and the other is $3x - 5$

Then
$$9x^2 - 25 = (3x + 5)(3x - 5)$$

Example 2. Factor $x^{2n} - 16y^{2n}$

Solution $x^{2n} - 16y^{2n} = (x^n + 4y^n)(x^n - 4y^n)$

[A] **EXERCISES**

Factor the following binomials

1. $y^4 - 4$	9. $4x^2 - 1$	17. $1 - 100x^2$
2 $x^2 - 9$	10 $9c^2 - 16$	18 $9 - 64c^4$
3. $m^2 - 1$	11 $x^2 - 25$	19. $25b^{2n} - a^2$
4. $1 - y^2$	12 $y^2 - 81$	20 $16a^2 - 49b^{4m}$
5 $c^2 - d^2$	13. $a^2b^2 - c^2$	21 $4a^{6n} - 81b^{2n}$
6 $x^2 - 49$	14. $x^2y^2 - m^2$	22 $y^{2m} - .01$
7. $x^6 - 25$	15 $9a^4b^2 - c^6$	23 $m^{4x} - n^{2y}$
8. $y^4 - 36$	16. $16x^2y^6 - 25$	24. $\frac{4}{9}a^2 - \frac{25}{36}b^2$

The Sum and Difference of Two Cubes [A]

Study the multiplication and division below

$$
\begin{array}{r}
a^2 - ab + b^2 \\
\underline{a + b} \\
a^3 - a^2b + ab^2 \\
\underline{a^2b - ab^2 + b^3} \\
a^3 + b^3
\end{array}
\qquad
\begin{array}{r}
a^2 - ab + b^2 \\
a + b\,\overline{\big)\,a^3 + 0 \quad + 0 \quad + b^3} \\
\underline{a^3 + a^2b} \\
-a^2b + 0 \\
\underline{-a^2b - ab^2} \\
ab^2 + b^3 \\
\underline{ab^2 + b^3}
\end{array}
$$

From either of these operations we know that

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

In like manner it can be shown that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

You should remember these formulas

$$\boxed{\begin{array}{c} a^3 + b^3 = (a + b)(a^2 - ab + b^2) \\ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \end{array}}$$

ORAL EXERCISES

[A]

These exercises refer to the two formulas above Supply the missing words

1. $a^3 - b^3$ is the difference of the ..?.. of a and b

2. $a^3 + b^3$ is the ..?.. of the ..?.. of a and b

3. The sum of the cubes of two numbers is divisible by the ..? of the numbers, and the difference of the cubes of two numbers is divisible by the ..?.. of the numbers

4 How can you remember the signs of ab in the second factors?

Example 1. Factor $8x^3 - 27$

Solution The cube root of $8x^3$ is $2x$ and the cube root of 27 is 3 Then $2x - 3$ is one factor The first term of the second factor is $(2x)^2$, or $4x^2$, the second term of this factor is $(2x)(3)$, or $6x$, and the third term of this factor is 3^2, or 9

Then $\qquad 8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)$

Example 2. Factor $y^3 + 64$

Solution $y^3 + 64 = (y + 4)(y^2 - 4y + 16)$

EXERCISES [A]

Factor

1 $a^3 - 1$	7 $m^3 - n^3$	13. $x^6 + y^6$	19. $a^3 + \frac{1}{8}$
2 $a^3 + 1$	8. $m^3 + n^3$	14. $y^6 + 8$	20. $b^3 - \frac{1}{27}$
3 $x^3 + 8$	9. $x^3 + 64$	15 $a^3 + 125$	21 $x^{3a} + 1$
4 $x^3 - 8$	10. $y^3 - 125$	16. $r^3 - 216$	22. $y^{3b} - 64$
5. $c^3 + 27$	11. $8x^3 - 1$	17 $a^3b^3 + 1$	23. $\frac{1}{8}a^3 - \frac{1}{27}b^3$
6. $m^3 - 27$	12. $8y^3 + 27$	18. $x^3y^3 + m^3$	24. $.001c^3 - .008d^3$

Complete Factoring [A]

In the preceding exercises you have factored five types of polynomials Of these the perfect-square trinomial and the difference of squares were special cases of the quadratic trinomial

From now on, when any expression is to be factored, it should be factored completely This means that any factor that is not prime should be factored Then all the factors are prime

Suggestions for Complete Factoring

1. Remove the largest monomial factor (if any)
2. If the polynomial is a binomial,
   a It may be the difference of two squares.
   b It may be the difference of two cubes
   c It may be the sum of two cubes
3. If the polynomial is a trinomial,
   it may be factored by the "guess" method
   A special case is the perfect square trinomial

Example 1. Factor $a^5 - 16a$

Solution. The largest monomial factor is a Then

$$a^5 - 16a = a(a^4 - 16)$$
$$= a(a^2 + 4)(a^2 - 4)$$
$$= a(a^2 + 4)(a + 2)(a - 2)$$

Example 2. Factor $6x^3 + 21x^2 - 45x$

Solution. $6x^3 + 21x^2 - 45x = 3x(2x^2 + 7x - 15)$
$$= 3x(2x - 3)(x + 5)$$

Example 3 Factor $x^6 - y^6$

Solution This expression can be factored as the difference of cubes or as the difference of squares In a case like this *always* factor first as the difference of squares Why? Then

$$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$$
$$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$$

EXERCISES

Factor into prime factors [A]

1 $4x + 12$
2 $a^2b + ab^2$
3 $m^2 - 9$
4 $4k^2 - 16$
5 $2a^2 - 2$
6 $3a^3 + 24$
7 $5m^2 - 30m - 30$
8 $4c^2 + 24c + 36$
9 $m^3 - 125$
10 $3m^3 + 192$
11 $cx^2 - 4c$
12 $4c^3 - 4c^2 + 4c$
13 $3s^2 - 24st + 48t^2$
14 $3x^2 - 15xy - 42y^2$
15 $6y^2 - 9yz - 15z^2$
16 $3ax^2 + 5ax + 2a$
17 $9bx^2 - 30bx + 24b$
18 $abx^3 - 27ab$
19 $9x^3 - 72$
20 $4a^5 + 20a^4 + 25a^3$
21 $10r^2 - 40R^2$
22 $2cb^2 - 2c^3$
23 $x^4 - a^4$
24 $x^8 - y^8$
25 $x^3 + x^2 + x$
26 $ab^3 - a^2b + ab$
27 $9c^3 - 18c^2d + 9cd^2$
28 $a^{2x} - b^{2y}$
29 $c^{2a} - d^{2b}$
30 $128 - 2a^3$
31 $108 - 4c^3$
32 $x^6 + y^6$

Factoring by Grouping[A]

Some polynomials may be factored by grouping their terms so that they are of the types which you have already studied We shall consider the polynomials which can be grouped,

1 With terms having a common binomial factor
2 As the difference of two squares
3 In the form of a quadratic trinomial

Common Binomial Factors[A]

Factoring out a binomial factor is similar to factoring out a monomial factor

Example 1 Factor $a(x-y)+b(x-y)$

Solution 1. Let $\qquad x-y=c$

Then $\qquad a(x-y)+b(x-y)=ac+bc$

$$=c(a+b)$$

Replacing c by $x-y$, $\quad c(a+b)=(x-y)(a+b)$

Then $\qquad a(x-y)+b(x-y)=(x-y)(a+b)$

Solution 2. $x-y$ is a divisor of $a(x-y)+b(x-y)$

The quotient is $a+b$ - - - - - - - - - $\rightarrow$

$$\begin{array}{r} a \qquad\quad +b \\ x-y\overline{)a(x-y)+b(x-y)} \end{array}$$

Then $\qquad a(x-y)+b(x-y)=(x-y)(a+b)$

Example 2 Factor $(x+2y)^2-3(x+2y)$

Solution $x+2y$ is a binomial divisor of $(x+2y)^2-3(x+2y)$

The quotient is $x+2y-3$ - - - - - - $\rightarrow$

$$\begin{array}{r} x+2y \quad -3 \\ x+2y\overline{)(x+2y)^2-3(x+2y)} \end{array}$$

Then $\quad (x+2y)^2-3(x+2y)=(x+2y)(x+2y-3)$

[A] **EXERCISES**

Factor into prime factors

1 $a(b+c)+b(b+c)$
2 $x(a-y)-c(a-y)$
3 $c(a-b)+d(a-b)$
4. $(x-1)h-2(x-1)$
5 $c^2(m-1)+d^2(m-1)$
6 $x^2(x-4)-y^2(x-4)$
7. $x^2(x-1)-4y^2(x-1)$
8. $m(a-2)-n(a-2)$
9 $2c(x-5)-c^2(x-5)$
10. $4(p+2)-y(p+2)$
11 $a(b+3)-c(b+3)$
12 $m(x^2-9)-(x^2-9)$
13 $y^2(x+2)-x^2(x+2)$
14 $c^2(c+3)-9(c+3)$
15 $h^2(k-4)-25(k-4)$
16 $(x^2-9)-y^2(x^2-9)$
17. $x^n(a+b)-y^n(a+b)$
18 $a^2(x^a-y^b)-c^2(x^a-y^b)$
19 $(x-y)^3+2(x-y)^2-(x-y)$
20 $a(m+n)-b(m+n)-c(m+n)$

In the preceding exercises the polynomials were grouped so that the binomial divisors could be easily determined. In most cases the polynomials are not so grouped.

Example 3 Factor $3x-3y-ax+ay$

Solution. This polynomial can be written as a binomial

$$3x-3y=3(x-y) \quad \text{and} \quad -ax+ay=-a(x-y)$$

Then $\qquad 3x-3y-ax+ay=3(x-y)-a(x-y)$

$$=(x-y)(3-a)$$

Do you know why $-ax+ay$ was not written $a(-x+y)$?

EXERCISES

[A]

Factor into prime factors

1. $cx + cy + bx + by$
2. $mx + my - nx - ny$
3. $c^3 - c - c^2 + 1$
4. $x^3 + x + x^2 + 1$
5. $3y^3 - 6y^2 - 3y + 6$
6. $5a^3 + 2a^2 - 15a - 6$
7. $a^2x - x - 3a^2 + 3$
8. $2n^2 - 18 - 9m + mn^2$
9. $x^3 + 2x^2 + 3x + 6$
10. $ay^2 - a - y^2 + 1$
11. $x^4 - 8x + x^3y - 8y$
12. $c^2xy - c^3 - x^2y + cx$

Can you factor the following?

13. $x^3 + y^3 + x^2 - y^2$
14. $a^3 + b^3 - 2(a + b)$
15. $m^3 - 8 + m^2 - 4$
16. $b^3 - c^3 - 4b + 4c$

Polynomials That Can Be Grouped as the Difference of Two Squares [A]

If a polynomial can be expressed as the difference of two squares, it can be factored like $x^2 - y^2$

Example 1 Factor $(x - y)^2 - 9$

Solution Let	$x - y = m$
Then	$(x - y)^2 - 9 = m^2 - 9$
Factoring	$m^2 - 9 = (m + 3)(m - 3)$
Replacing m by $x - y$	$(m + 3)(m - 3) = (x - y + 3)(x - y - 3)$
Then	$(x - y)^2 - 9 = (x - y + 3)(x - y - 3)$

EXERCISES

[A]

Factor

1. $(x + y)^2 - c^2$
2. $(m - n)^2 - p^2$
3. $(a - b)^2 - 4d^2$
4. $(a + b)^2 - d^2$
5. $(b - 4)^2 - a^6$
6. $x^2 - (y + 1)^2$
7. $a^2 - (b + 2)^2$
8. $a^2 - (m - 1)^2$
9. $4h^2 - (k - 3)^2$
10. $9a^2 - (a + 3b)^2$

Example 2 Factor $c^2 - 9 - 2cd + d^2$

Solution. Three of these terms form the perfect square trinomial $c^2 - 2cd + d^2$ The remaining term, -9, with its sign changed is the square of 3

Then
$$\begin{aligned} c^2 - 9 - 2cd + d^2 &= (c^2 - 2cd + d^2) - 9 \\ &= (c - d)^2 - 3^2 \\ &= [(c - d) + 3][(c - d) - 3] \\ &= [c - d + 3][c - d - 3] \end{aligned}$$

Example 3. Factor $4x^2 - 4y^2 + 4y - 1$

Solution If the signs of the terms of $-4y^2 + 4y - 1$ are changed, the trinomial is a perfect square, $4y^2 - 4y + 1$

$$\begin{aligned} 4x^2 - 4y^2 + 4y - 1 &= 4x^2 - (4y^2 - 4y + 1) \\ &= 4x^2 - (2y - 1)^2 \\ &= [2x + (2y - 1)][2x - (2y - 1)] \\ &= [2x + 2y - 1][2x - 2y + 1] \end{aligned}$$

DO

EXERCISES

[A]

Factor

1. $x^2 - 2xy - 1 + y^2$
2. $a^2 + 2ab + b^2 - c^2$
3. $x^2 - 6x + 9 - 4y^2$
4. $y^2 - 4y + 4 - 25z^2$
5. $c^2 - 10c + 25 - 36d^2$
6. $1 - x^2 - 2xy - y^2$
7. $4 - a^2 - 2ab - b^2$
8. $x^2 - c^2 - y^2 + 2cy$
9. $m^2 + 2np - p^2 - n^2$
10. $1 - m^2 - 25n^2 + 10mn$

[B]

11. $c^2 - 2cd + d^2 - f^2 + 2fg - g^2$
12. $y^2 - 6y + 9 - x^2 - 10x - 25$
13. $m^2 - h^2 + n^2 - 2mn + 2hk - k^2$
14. $r^2 + t^2 - 4m^2 + 4m - 1 - 2rt$

Polynomials That Can Be Grouped Into the Form $ax^2 + bx + c$ [A]

Example Factor $(x + y)^2 + 3(x + y) - 28$

Solution Let $x + y = m$

Then $(x+y)^2 + 3(x+y) - 28 = m^2 + 3m - 28$

$= (m + 7)(m - 4)$

Replacing m by $x + y$ $= (x + y + 7)(x + y - 4)$

EXERCISES

[A]

Factor

1. $(a+b)^2 + 7(a+b) + 12$
2. $(x+y)^2 + 2(x+y) + 1$
3. $(m-n)^2 - 5(m-n) + 6$
4. $(x-2y)^2 - 3(x-2y) - 18$
5. $(c+d)^2 - 4(c+d) - 21$
6. $(y-1)^2 - 8(y-1) + 12$
7. $3(x-y)^2 - 14(x-y) + 8$
8. $4(y+3)^2 + 19(y+3) - 5$
9. $a^2 - 2a(b+c) + (b+c)^2$
10. $x^2 - x(x+y) - 20(x+y)^2$

Polynomials That Can Be Changed into the Difference of Two Squares [8]

Binomials and trinomials which have the forms $x^4 + 4y^4$ and $x^4 - 7x^2y^2 + y^4$ can often be factored by changing them into the difference of two squares. Study the two examples.

Example 1 Factor $a^4 + 4b^4$.

Solution a^4 and $4b^4$ are perfect squares. If $4a^2b^2$ is added to $a^4 + 4b^4$, we obtain $a^4 + 4a^2b^2 + 4b^4$, which is a perfect square trinomial. By adding $4a^2b^2$ to $a^4 + 4b^4$ and then subtracting it, as shown at the right, we obtain $a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2$, which may be expressed as the difference of two squares.

$$\begin{array}{r}
a^4 \qquad\qquad + 4b^4 \qquad\quad \\
4a^2b^2 \qquad\qquad - 4a^2b^2 \\
\hline
a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2
\end{array}$$

$$\begin{aligned}
a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\
&= (a^2 + 2b^2)^2 - (2ab)^2 \\
&= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)
\end{aligned}$$

Example 2 Factor $a^4 - 7a^2b^2 + b^4$.

Solution $a^4 - 7a^2b^2 + b^4$ can be made into a perfect square trinomial by changing $-7a^2b^2$ to either $+2a^2b^2$ or $-2a^2b^2$. Let us try adding and subtracting $5a^2b^2$ to the trinomial.

$$\begin{array}{r}
a^4 - 7a^2b^2 + b^4 \qquad\quad \\
5a^2b^2 \qquad\quad - 5a^2b^2 \\
\hline
a^4 - 2a^2b^2 + b^4 - 5a^2b^2
\end{array}$$

The first three terms of the sum form a perfect square, but $5a^2b^2$ is not a perfect square. Let us try adding and subtracting $9a^2b^2$ as shown.

$$\begin{array}{r}
a^4 - 7a^2b^2 + b^4 \qquad\quad \\
9a^2b^2 \qquad\quad - 9a^2b^2 \\
\hline
a^4 + 2a^2b^2 + b^4 - 9a^2b^2
\end{array}$$

This result may be expressed as the difference of the two squares $(a^2 + b^2)^2$ and $(3ab)^2$.

$$\begin{aligned}
a^4 - 7a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - 9a^2b^2 \\
&= (a^2 + b^2)^2 - (3ab)^2 \\
&= (a^2 + b^2 + 3ab)(a^2 + b^2 - 3ab)
\end{aligned}$$

[8]

EXERCISES

Factor.

1 $x^4 + 4y^4$	5 $a^4 + a^2b^2 + b^4$	9 $9c^4 + 5c^2 + 1$
2 $4x^4 + y^4$	6 $x^4 + x^2y^2 + y^4$	10 $4x^4 + 4x^2 + 25$
3 $a^4 + a^2 + 1$	7 $4x^4 + 3x^2 + 1$	11 $4x^4 - 29x^2 + 25$
4 $c^4 + c^2 + 1$	8 $y^4 - 7y^2 + 1$	12 $y^4 - 17y^2 + 64$

The Remainder and Factor Theorems[M]

Two theorems, the remainder theorem and the factor theorem, enable us to factor some polynomials which do not yield to any of the factoring methods mentioned thus far. The proofs for these theorems are given on page 524, but for the present we shall content ourselves with a study of what the theorems mean and how they can be used.

The polynomial $x^2 - 3x - 18$ is a function of x, since its value depends upon the value of x. The words "function of x" may be abbreviated $f(x)$, which is often read "f of x."

If $f(x) = x^2 - 3x - 18$, then $f(2)$, the value of the function when $x = 2$, is found by substituting 2 for x in the polynomial, and $f(-2)$ is found by substituting -2 for x in the polynomial.

Let us now make a study of some divisions and evaluations.

$$f(x) = x^2 - 3x - 18$$

$$(x^2 - 3x - 18) \div (x + 2) = x - 5 + \frac{-8}{x+2}, \qquad f(-2) = 4 + 6 - 18 = -8$$

$$(x^2 - 3x - 18) \div (x - 2) = x - 1 + \frac{-20}{x-2}, \qquad f(2) = 4 - 6 - 18 = -20$$

$$(x^2 - 3x - 18) \div (x + 3) = x - 6 + \frac{0}{x+3}, \qquad f(-3) = 9 + 9 - 18 = 0$$

Compare the three remainders in the divisions with the three values of the polynomial. Can you tell how to find the remainder when $x^2 - 3x - 18$ is divided by $x + 6$? by $x - 6$? by $x + 10$? These divisions and evaluations illustrate the remainder theorem.

The Remainder Theorem

a polynomial in x is divided by the binomial $x - a$, the remainder is the same as the result obtained when $+a$ is substituted for x in the polynomial. Or, if $f(x)$ is divided by $x - a$, remainder $= f(a)$

Study the division $(x^2 - 3x - 18) \div (x + 3)$ above. Notice that since there is no remainder, $x + 3$ is a factor of $x^2 - 3x - 18$. Notice that neither of the other divisors listed are factors because they are not exact divisors. This thinking brings us to the factor theorem.

The Factor Theorem

If a polynomial in x equals zero when a is substituted for x, then $x - a$ is a factor of the polynomial
In other words, if $f(a) = 0$, $x - a$ is a factor of $f(x)$

Let us now see how the factor theorem can aid us in factoring. Suppose we wish to factor $x^3 - 5x + 6$. If $x^3 - 5x + 6$ has a factor of the type $x - a$, then a is an exact divisor of 6. The possibilities for a are $+1$, -1, $+2$, -2, $+3$, -3, $+6$, and -6. To find which of these will make the value of the polynomial zero, we make use of the factor theorem.

$f(x) = x^3 - 5x^2 + 6$
$f(1) = 1 - 5 + 6 = 2$
$f(-1) = -1 - 5 + 6 = 0$. Then $x + 1$ is an exact divisor of $x^3 - 5x^2 + 6$.

By division, $x^3 - 5x^2 + 6 = (x + 1)(x^2 - 6x + 6)$. The trinomial $x^2 - 6x + 6$ is prime.

Then $x^3 - 5x^2 + 6 = (x + 1)(x^2 - 6x + 6)$.

Example Factor $x^5 + 32$

Solution

$$f(x) = x^5 + 32$$
$$f(1) = 1 + 32 = 33$$
$$f(-1) = -1 + 32 = 31$$
$$f(2) = 32 + 32 = 64$$
$$f(-2) = -32 + 32 = 0$$

Then $x + 2$ is an exact divisor of $x^5 + 32$.

By division, $x^5 + 32 = (x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$

[A]

EXERCISES

Factor, using the factor theorem.

1. $x^3 + x^2 + 4$
2. $x^3 + 4x - 5$
3. $x^3 - 2x^2 + 3$
4. $x^3 - 2x - 4$
5. $2x^3 - x - 1$
6. $x^5 + y^5$
7. $a^5 - y^5$
8. $2x^3 - 54$
9. $x^2 - x - 2$
10. $x^2 - 9$
11. $x^2 + 2x - 3$
12. $x^2 + 4x - 5$
13. $x^3 - 2x^2 - 19x + 20$
14. $x^7 + y^7$

Directions for Complete Factoring (4)

We shall now revise the directions for factoring found on page 77

Directions for Complete Factoring

A. Remove the largest monomial factor, if any

B. To factor any polynomial, count the number of its terms.

1. If it is a binomial,
   - a. It may be the difference of two squares
   - b. It may be the difference of two cubes
   - c. It may be the sum of two cubes
   - d. By the addition and subtraction of the same monomial, the binomial may be changed into the difference of two squares
2. If it is a trinomial,
   - a. It may be a general quadratic of the form $ax^2 + bxy + cy^2$, of which a perfect-square trinomial is a special type
   - b. By the addition and subtraction of the same monomial, the trinomial may be changed into the difference of two squares
3. If it is a polynomial of four or more terms,
   - a. It may be grouped to show a common binomial factor
   - b. It may be grouped as the difference of two squares
   - c. It may be grouped in the form of a quadratic trinomial

C. It may be factored by the use of the factor theorem

EXERCISES

Factor into prime factors

1. $x^2 - 7x$
2. $x^3 + 9x$
3. $x^3 - 9x$
4. $5x^2 + 35x + 60$
5. $11p^2 - 11p - 66$
6. $4c^4d - 16c^2d^5$
7. $40x^3 - 5y^3$
8. $x^2 - (a - b)^2$ $p^2 - p - 6$
9. $c^2 - (m + n)^2$
10. $x^2 + 2xy + y^2 - c^2$
11. $a^3 + a + a^2 + 1$
12. $a^2c^2 - 4a^2 - b^2c^2 + 4b^2$
13. $2x^3y - 8xy^3$
14. $3a^4b + 3ab^4$
15. $x^3 + x^2 - 2$
16. $a^{2m} - b^{2n}$

17 $x^{3a} + y^{3a}$

18 $r^4 - 4s^4$

19 $b^6 - c^6$

20 $4x^4 - 4$

21 $5x^2 - 405$

22 $3y^3 - 6y^2 - 3y + 6$

23 $2a^3 - 6a^2 - 2a + 6$

24 $8m^2 - 11m + 3$

25 $14n^2 - 10n - 4$

26 $a - a^2 - a^3 + a^4$

27 $a^3 - 4a^2 - a + 4$

28 $16 - x^8$

29 $(x+3)^2 - (x+3) - 56$

30 $(c-1)^2 - 2(c-1) - 35$

31 $a^2 - b^2 + (a-b)^2$

32 $c^2 + cd - 5c - 5d$

33 $5ac - 5ad + 5bc - 5bd$

34 $a^2x - x + a^2y - y$

35 $a^2b^2 + ab^3 - ab^2$

36 $m^3n - m^2n^2 + m^2np$

37 $x^{2n} - 5x^n - 36$

38 $a^{2x} + 4a^x - 21$

39 $27x^3 + 64y^3$

40 $4 - 500y^3$

[B]

Factor

41 $x^4 - 9x^2 + 16$

42 $y^4 - 19y^2 + 9$

43 $x^4 - 14x^2 + 1$

44 $x^4 + 4$

45 $a^3 - 1$

46 $a^5 + 1$

47 $x^3 - 3x^2 + 3x - 2$

48 $y^4 - 81$

Lowest Common Multiple [A]

The lowest common multiple (L C M) of two or more numbers is the expression with the least number of factors which will exactly contain each of the given numbers It is the expression having the lowest degree and smallest numerical coefficient that can be divided by each of the numbers and have integral quotients Thus 12 is the L C M of 2 4 and 6 x^5 is the L C M of x^3, x^5, and x and $15ac$ is the L C M of $3c$ $15c$, and $5a$

To Find the L C M of Two or More Expressions

1 Find the prime factors of each expression

2 Then find the product of all the different factors using each factor the greatest number of times it occurs in any one of the given expressions.

Example 1 Find the lowest common multiple of $4a^3$, $7a^2b$, and $2ab$

Solution The lowest common multiple of 4, 7, and 2 is 4×7, or 28 The lowest common multiple of a^3, a^2, and a is a^3 The lowest common multiple of b and b is b Then the lowest common multiple of $4a^3$, $7a^2b$, and $2ab$ is $28a^3b$

In this example the lowest common multiple was found by inspection

Example 2 Find the L C M of $2a^3 - 18a$, $4a^2 + 6a - 18$, and $a^2 - 6a + 9$

Solution We shall use the rule to find the lowest common multiple of these polynomials

$$2a^3 - 18a = 2a(a+3)(a-3)$$
$$4a^2 + 6a - 18 = 2(a+3)(2a-3)$$
$$a^2 - 6a + 9 = (a-3)^2$$

By the rule above, the factor 2 should be used once as a factor of the L C M , the factor a should be used once, the factor $a+3$ should be used once, the factor $2a-3$ should be used once, and the factor $a-3$ should be used twice

Then the L C M $= 2a(a+3)(2a-3)(a-3)^2$

(A) **EXERCISES**

Find the lowest common multiple of

1 20, 30, and 45
2. 18, 27, and 36
3. $8xy$, $12x^2$, and $18y^2$
4 $15a^2bc^3$ and $20a^3bc^4$
5 $(a+b)(c-d)^2$ and $(m+n)(c-d)$
6 x^2-y^2, $2x+2y$, and $x^2+2xy+y^2$
7 a^2-b^2, a^3-ab^2, and a^2+ab^2
8 x^2+3x, x^2-9, and x^2-3x
9 $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$
10 $(x+y)^3$, $2(x+y)^2$, and $3(x+y)$
11 m^2-5m+6 and m^2-4m+4
12 c^2-16, c^2-c-12, and c^2-9
13 x^3+y^3, $x+y$, and x^2-xy+y^2
14 a^3-b^3, $a-b$, and $2a^2+2ab+2b^2$
15 $2x^2+x-3$ and x^3-x^2-x+1
16 a^3+a^2-a-1 and a^2-2a+1

Equations That Contain Type Products [A]

Equations that contain type products can be solved more quickly because the multiplications can be performed mentally

Example Solve $(2x-1)^2-(x+1)^2=3(x+2)(x-2)$

Solution The multiplications should be done mentally, and the products should be placed in parentheses

$$
\begin{aligned}
(2x-1)^2-(x+1)^2 &= 3(x+2)(x-2)\\
(4x^2-4x+1)-(x^2+2x+1) &= 3(x^2-4)\\
4x^2-4x+1-x^2-2x-1 &= 3x^2-12\\
3x^2-6x &= 3x^2-12\\
-6x &= -12\\
x &= 2
\end{aligned}
$$

PROOF Does $(4-1)^2-(2+1)^2=3(4)(0)$?
Does $9-9=0$? Yes

[A]

EXERCISES

Solve and check

1 $(2x-1)^2-(x-3)(3x+2)=x^2-5$

2 $(x+6)(2x+3)-(x+5)(2x-1)=11$

3 $(m+6)(m-6)-m(m-5)=29$

4 $(p+3)(p-1)-(p+2)(p-2)=0$

5 $(x+4)(x-5)+(x+2)(x-6)=2x^2-2$

6 $(2x-1)(x-3)-3(x-1)(x+2)=29-x^2$

7 $(x-4)(2x+1)+(x+1)(3x-2)=5x^2-30$

8 $(x+1)(x^2-x+1)+3x=x^3-5$

Literal Equations [A]

The solutions of the literal equations given in Chapter 2 did not involve factoring We shall now solve literal equations which require a knowledge of factoring

Example 1 Solve $ax+b^2=a^2+bx$ for x

Solution	$ax+b^2=a^2+bx$
	$ax-bx=a^2-b^2$
Factoring	$(a-b)x=(a+b)(a-b)$
D[illegible]	$x=a+b$

PROOF Does $a(a+b)+b^2=a^2+b(a+b)$? Yes

Example 2. Solve for x $b(x-b)=x-(2-b)$

Solution	$b(x-b)=x-(2-b)$
Removing parentheses	$bx-b^2=x-2+b$
A_{b^2}, S_x	$bx-x=b^2+b-2$
Factoring	$x(b-1)=(b+2)(b-1)$
D_{b-1}	$x=b+2$

PROOF Left to student

[A] EXERCISES

Solve for x or y

1 $ax-bx=a^2-2ab+b^2$
2 $mx+4=x+4m^2$
3 $(c-d)y=c^2-2cd+d^2$
4 $cx+x=(c+1)(c-1)$
5 $bx-b^2=3x+b-12$
6 $cy-1=c^2+2c-y$
7 $m(m-x)=n(n-x)$
8 $b^2(ay-1)=a^2(1-ay)$
9 $a^2-ax-b^2=bx$
10 $b^2-a^2=ab^2y-a^3y$
11 $a^3-bx=ax-b^3$
12 $ax-8=a^3-2x$

Solving Formulas [A]

The formula $A=p+prt$ tells how to find the amount when the principal, rate, and time are known In this formula A is the subject and "$=p+prt$" is the predicate Let us solve the formula for p, that is, make p the subject

	$A=p+prt$
	$-p-prt=-A$
D_{-1}	$p+prt=A$
Factoring,	$p(1+rt)=A$
D_{1+rt}	$p=\dfrac{A}{1+rt}$

PROOF Left to student

This formula tells how to find the principal when the amount, rate, and time are known

[A] EXERCISES

1 Solve $A=p+prt$ for t, and state the rule for finding the time when the amount, principal, and rate are given

2 Solve the formula $A=p+prt$ for r, and tell how to find the rate when the amount, principal, and time are known

3 Solve $A=bh$ for b, for h

4 Solve $A = \frac{1}{2}bh$ for b, for h

5 $pv = c$ expresses the relation between the pressure and volume of a confined gas Solve the formula for p, for v

6 $S = 4\pi r^2$ is a formula for finding the area of a sphere Solve it for r^2 If you know the value of r^2, how can you find the value of r?

7 $A = \frac{1}{2}h(b + b')$ is a formula for finding the area of a trapezoid Solve it for h and change the resulting formula into a rule

8 $C = \pi D$ How can you find the value of π when you know the values of C and D?

9 Solve $T = mg - mf$ for f

10 Solve $T = mg - mf$ for m

11 Solve $2A = h(b + c)$ for h

12 Solve $A + B + C = 180$ for A

13 $A = \pi R^2 - \pi r^2$ is a formula for finding the area of a circular ring

Factor the right member of the formula If you know the values of A, π, and $R - r$, how can you find the value of $R + r$?

14 $S = \pi rl + \pi r'l$ is a formula for finding the lateral area of a frustum of a cone of revolution Solve it for l, the slant height of the frustum

15 Solve $T = 2\pi r^2 + 2\pi rh$ for h

16 Solve for s $r(s - l) = s - a$

Solving Quadratic Equations by Factoring[A]

Factoring is useful in solving some second degree equations in one unknown A second-degree equation in one unknown is an equation which contains the second power of the unknown and no power of the unknown higher than the second Second degree equations are called quadratic equations

If a quadratic equation can be reduced to the form $f(x) = 0$, where $f(x)$ is factorable, the equation can be solved by factoring The solution depends upon the following fact

If the indicated product of two or more factors is zero, at least one of the factors is equal to zero

For example, if $3\,xy = 0$, then $x = 0$, $y = 0$, or both x and y equal 0. The third factor, 3, obviously cannot be zero. If $x = 0$, then $3 \cdot 0 \cdot y = 0$ for all values of y. If $y = 0$, then $3 \cdot x \cdot 0 = 0$ for all values of x. If both $x = 0$ and $y = 0$, then $3 \cdot 0 \cdot 0 = 0$.

Example 1 Solve $6\,x^2 + 15\,x = 9$

Solution

$$6\,x^2 + 15\,x = 9$$
$$6\,x^2 + 15\,x - 9 = 0$$
$$3(2\,x^2 + 5\,x - 3) = 0$$
$$3(2\,x - 1)(x + 3) = 0$$

If $2\,x - 1 = 0$	If $x + 3 = 0$
$2\,x = 1$	$x = -3$
$x = \frac{1}{2}$	
PROOF that $\frac{1}{2}$ is a root	PROOF that -3 is a root
Does $6(\frac{1}{2})^2 + 15(\frac{1}{2}) = 9$?	Does $6(-3)^2 + 15(-3) = 9$?
Does $\frac{3}{2} + \frac{15}{2} = 9$? Yes	Does $54 - 45 = 9$? Yes

Example 2 Solve $x^2 = 3\,x$

Solution $x^2 = 3\,x$

Then $x^2 - 3\,x = 0$

Factoring $x(x - 3) = 0$

	If $x - 3 = 0$
$x = 0$	$x = 3$
PROOF that 0 is a root	PROOF that 3 is a root
Does $0^2 = 3 \cdot 0$? Yes	Does $3^2 = 3 \cdot 3$? Yes

Rule for Solving Quadratic Equations by Factoring

1. Reduce the equation to the form $ax^2 + bx + c = 0$
2. Factor the left member
3. Set each factor that contains the unknown equal to zero
4. Solve each linear equation obtained in step 3
5. Prove that each value found is a root

EXERCISES

[A]

Solve by factoring

x=-6 10

1 $x^2 - 10x + 24 = 0$

2 $y^2 + 8y + 7 = 0$

3 $c^2 + 5c = 24$

4 $x^2 = 3x - 2$

5 $x^2 + 5 = 6x$

6 $m^2 - 2m - 15 = 0$

7 $y - 6 = -y^2$

8 $15y^2 = y + 6$

9 $2y^2 + 5 = 11y$

10 $y^2 + 12y = -32$

11 $80 + 2x = x^2$

12 $y(y - 7) = 44$ -9

13 $x(x + 11) + 18 = 0$

14 $4x^2 + 49 = -28x$

15 $3c^2 - 6c + 3 = 0$ c=1

16 $x^2 - 36 = 0$

17 $4c^2 - 36 = 0$ c=3 -3

18 $y^2 + 8y = 0$

19 $2c^2 - 20c = -18$ c=1 9

20 $7x^2 + 7 = 50x$

Solve for x or y

21 $8y^2 + 5cy - 3c^2 = 0$

22 $2y^2 - 5ay - 7a^2 = 0$

23 $x^2 + 7bx = -10b^2$

x=-2b

x=-5b

24 $5x^2 - 2a^2 = 3ax$

25 $2(3x^2 - 1) - 5x = 6x$

26 $4(y^2 + y) + 3 = -4y$

Checking Your Understanding of Chapter 3

Chapter 3 was designed to help you develop skill in multiplying and factoring. At this point you should check your ability to carry out the following processes easily.

Check yourself.

1 You should be able to multiply

	PAGE
A polynomial by a monomial	65
Two binomials	65–68

This includes ability to recognize the special product which results from squaring a binomial (p. 67) and the special product which results from multiplying the sum and difference of two quantities (p. 68).

Two polynomials which can be treated as two binomials	68

2 You should be able to factor a

Polynomial containing a monomial factor	70
Trinomial which can be factored	71
Trinomial which is a perfect square	74
Binomial which is the difference of two squares	75
Binomial which is the sum or difference of two cubes	76
Polynomial containing a binomial factor	78
Polynomial whose terms can be grouped to form the difference of two squares	80
or changed to form such a difference	81
Polynomial whose terms can be grouped to form a factorable quadratic trinomial	81
Polynomial which lends itself to use of the factor theorem	84

3 You should be able to find the lowest common multiple of two or more numbers 86

4 You should be able to solve

Linear equations which contain type products	87
Literal equations which involve factoring	88
Quadratic equations which can be reduced to the form $f(x) = 0$ where $f(x)$ is factorable	90

5 You should know the meaning of the following phrases and be able to spell the words in them

MATHEMATICAL VOCABULARY

lowest common multiple (p 86)
prime factor (p 71)
quadratic equation (p 90)
second-degree equation (p 90)

CHAPTER REVIEW

[A]

1 If one number is divided by another, the divisor and quotient are ? of the dividend

2 Is $a + b$ a factor of $a^3 - 2ab^2 - b^3$?

3 Is $m - 2n$ a factor of $m^3 - n^2 - 3m - 2$?

Find the following products mentally if you can

4 $\pi(R^2 - \pi R)$

5 $\frac{1}{2}x(4x^2 - 8)$

6 $(2c - d)(c + 3d)$

7 $(x - ab)(x + ab)$

8 $(a^m + b^n)(a^m - 2b^n)$

9 $(x^a - y^b)(x^a + y^b)$

10 $(x^2 - 2x + 4)(x + 2)$

11 $(a^2 + 2ab + 4b^2)(a - 2b)$

12 $(a - b + c)(a - b - c)$

13 $(x + 2y - z)(x - 2y + z)$

Factor into prime factors

14 $a^3 - 16a$

15 $27b^2 - 3c^2$

16 $27m^3 - n^3$

17 $(x + y)^2 - 4$

18 $cx + cy - nx - ny$

19 $p^4 - 29p^2 + 100$

20 $x^4 + 3x^2 - 28$

21 $c^{2m} - d^{2n}$

22 $c^2 + 6c - 16x^2 + 9$

23 $x^2 - y^2 - 4x + 4$

24 $4x^3 - 5x^2 - 6x$

25 $4c^2y^2 - (a - b)^2$

26 $1 - 8a^3b^3$

27 $m^6 - 1$

28 $d^3 - 3d^2 - 4d + 12$

29 $(m - n)^2 + 8(m - n) + 15$

30 Show by the remainder theorem that $x + 2$ is a factor of $x^3 + 4x^2 + x - 6$

What are the prime factors of $x^3 + 4x^2 + x - 6$?

31 Show by the factor theorem that $x + y$ is an exact divisor of $x^3 + 3x^2y + 3xy^2 + y^3$

Solve for x

32 $(2x - 1)(x + 5) + (x + 3)(3x - 1) = 5x^2 + 26$

33 $(3x - 1)^2 - (x + 5)(x - 4) = 8x^2 - 7$

34 $mx + l = ml + x$

35 Solve for b $\quad c^2 - 6b = 3c - 2bc$

36 Solve for s $\quad as + a = ar^n + rs$

37 Solve for x $\quad a(x - a) + b(x - b) = b(b + x) - (a + b)$

38. Square the binomials as indicated

a. $(x-y)^2$ c. $(2m-3n)^2$

b. $(a-2b)^2$ d. $(x-\frac{1}{3})^2$

39. Tell which of the following trinomials are perfect squares, and give the principal square roots of the perfect squares

a. h^2-6h+9 d. m^2-8m+4

b. $c^2-10c+25$ e. $y^2-4y+16$

c. $4p^2+4p+1$ f. $9x^2-30x+25$

40. $f(x)=2x^2-5x+6$ Which is the greater, and by how much, $f(2)$ or $f(3)$?

41. Solve by factoring

a. $x^2-4x-21=0$ c. $y^2-4y=0$

b. $m^2-13m=-36$ d. $2x^2-5x=12$

[8]

42. Factor

a. $m^2-6pm+9p^2-n^2$ d. x^6-64y^6

b. $x^2-4z^2+y^2-2xy$ e. x^4-22x^2+21

c. $y^3-3y^2-10y+24$ f. $r^3-3r^2-9r+27$

CUMULATIVE REVIEW

1 Simplify $(x - y)(x^2 + xy + y^2) - (x + y)(x^2 - xy + y^2)$

2 Factor

a $3y^2 + 4y - 4$

b $27 - y^3$

c $a^5 - a$

d $mx + my - nx - ny$

3 Find the value of $x^2y - (xy)^2$ when $x = 3$ and $y = -2$

4 Find the total area of a rectangular solid whose length is $2a - b$ width $a + b$ and height $3a$

5 Find the volume of the rectangular solid in Exercise 4

6 Place all terms except the first inside parentheses preceded by a minus sign $x^2 - a^2 + ab - b^2$

7 $f(x) = 4x^2 - 6x + 3$ Find $f(-1)$

8 If $f(x)$ is divisible by $x - a$, what is the value of $f(a)$?

9 Multiply $x^{2a} + x^a - 1$ by $x^a + 1$

Solve

10 $3(x + 4) - 2(7 - x) = 0$

11 $(2x - 1)^2 - 10 = 4x^2$

12 $10.6y = 3.8y - 1.36$

13 $0.09y - 12.8 = 13$

14 $(2x - 1)(3x + 2) - (x^2 - x) = 5x^2$

15 $4(x - 2)^2 - (x - 3)(3x + 2) - x^2 + 7 = 0$

16 Frank has five times as much money as Bill If Frank should give Bill \$20, he would have \$21 more than twice as much as Bill How much money does each have?

17 How much water must be added to 10 gallons of a 4% salt solution to make a 3% solution?

18 Show that -2 is a root of $x^3 + 3x^2 + 3x + 2 = 0$

19 Solve $S = 180(n - 2)$ for n

20 A salesman drove 90 miles going half the distance at 30 miles an hour and half the distance at 60 miles an hour If instead he had driven half the time of the trip at 30 miles an hour and half the time at 60 miles an hour, how much driving time would he have gained or lost?

[A] CHAPTER TEST

Part I

Find the products

1 $x^2(2x - 1)$

2 $(2x - 1)(x + 3)$

3 $(m^2 - 1)(m^2 + 1)$

4 $(2c + 3)(c - 5)$

5 $(2x - y^2)(x - 3y^2)$

6 $(x^n + 3)^2$

7 $(a^{2c} - b^{2d})(a^{2c} + b^{2d})$

8 $(a + b - 3c)^2$

9 $[(x - y) - 6][(x - y) + 2]$

10 $4(7c + 6)(2c - 1)$

Part II

Factor into prime factors

1 $m^2 - 64$

2 $ab^2 - 2a^2b$

3 $4x^2 + 12xy + 9y^2$

4 $1 - 81y^2$

5 $k^3 + 8$

6 $c^4 - 13c^2 + 36$

7 $(a - b)^2 - m^2$

8 $h^3 - 27$

9 $a^3 - 2a^2 + 3a - 6$

10 $5x^2y + 20xy + 20y$

11 $a^4 + 5a^2 + 4$

12 $1 - 6p + 9p^2$

13 $m^2 - n^2 + 10n - 25$

14 $48x^2 + 36x + 6$

15 $c^4 - d^4$

16 $ax^4 + 2ax^2 - 24a$

17 $(a + 1)^2 - 3(a + 1) - 18$

18 $x^3 - 2x + 1$

Part III

Solve for x

1 $(x - 2)(2x + 1) - (x - 3)^2 = x^2 - 17$

2 $ax + 6a = -2x - 3a^2$

3 Solve for p $A - p = prt$

4 $3x^2 - 21x + 36 = 0$

Mathematics and the Telephone

Research engineers and field men are continually trying to increase the ease and efficiency of telephone communication. In recent years they have largely eliminated interference and distortion; they have made great improvements in long distance communication through the use of amplifiers; and they have learned how to increase the number of communications by using different frequencies on the same circuit and by enclosing a large number of two wire and four wire circuits in cables.

The engineers of a telephone system are highly trained in the science of electricity, particularly in the branch known as communications. Their work requires two or more years of college mathematics, including college algebra, trigonometry, analytic geometry, and calculus.

The remainder of this article discusses in detail a simple problem of the type that telephone field men constantly have to deal with and suggests the service rendered by mathematics in this kind of work.

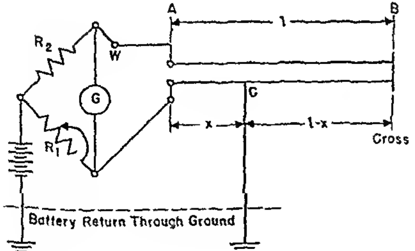
During a storm a defect occurred in a two wire circuit joining points A and B, which were 60 miles apart. The problem was to locate the trouble.

Brown Brothers

One method of locating a defect of this kind is the Murray Loop Test, a diagram of which is shown below The formula $\frac{x}{2l - x} = \frac{R_1}{R_2}$ gives the relationship between the resistances R_1 and R_2, measured in ohms, and the distances l and x, when the resistances of the two wires joining A and B are equal The distance, represented by x in the formula, from the point A to the defect in the wire can be determined by computation when the values of l, R_1, and R_2 are known

In this case the engineer sent one of his men to make a cross of the wires at B He connected a Wheatstone bridge to the two wires at A He used 1000 ohms resistance for R_2 and adjusted the resistance R_1 to balance the bridge The bridge balanced when R_1 was 146 ohms. By substituting known values in the formula given above, he obtained the equation $\frac{x}{120 - x} = \frac{146}{1000}$ Solving for x, he found that the defect was 15 29 miles from A

Ordinarily the procedure is much more complicated For example, the wires do not often have uniform resistance



CHAPTER 4

Fractions and Fractional Equations

In this chapter you will study operations with fractions and solve equations containing fractions

An Important Principle[A]

In order to solve many problems occurring in applied science and other fields, it is necessary to use fractions. A fraction is an indicated quotient of two numbers or expressions.

Fundamental Principle of Fractions

Both terms of a fraction may be multiplied or divided by the same number (not including zero) without changing the value of the fraction.

Two fractions are said to be equivalent when one fraction can be changed to the other by either multiplying or dividing both terms of the fraction by the same number.

Signs of a Fraction[A]

A fraction has three signs associated with it: (1) the sign of the fraction, (2) the sign of the numerator, and (3) the sign of the denominator.

If we multiply both numerator and denominator of a fraction by -1, both terms of the fraction have their signs changed but the value of the fraction is unchanged. If we multiply the fraction by -1 and its numerator by -1, the sign of the fraction and the sign of the numerator are changed but the value of the fraction is unchanged, and if we multiply the fraction by -1 and the denominator by -1, the sign of the fraction and the sign of the denominator are changed but the value of the fraction is unchanged. Then we may state:

Any two of the three signs associated with a fraction may be changed without changing the value of the fraction.

See how this rule applies to the fraction $\frac{6}{2}$, whose value is 3.

CHANGE TO (numerator), CHANGE TO (denominator):

$$+\frac{+6}{+2} \qquad +\frac{-6}{-2} \qquad\qquad +\frac{-6}{-2} = +(+3) = 3$$

CHANGE TO (numerator), CHANGE TO (sign of fraction):

$$+\frac{+6}{+2} \qquad -\frac{-6}{+2} \qquad\qquad -\frac{-6}{+2} = -(-3) = 3$$

CHANGE TO (sign of fraction), CHANGE TO (denominator):

$$+\frac{+6}{+2} \qquad -\frac{+6}{-2} \qquad\qquad -\frac{+6}{-2} = -(-3) = 3$$

If the numerator (or denominator) is a polynomial, its sign is changed by changing the signs of all its terms If in the fraction $\frac{x+y-z}{x-3}$ we change the sign of the numerator and the sign of the fraction, we have

$$\frac{x+y-z}{x-3} = -\frac{-(x+y-z)}{x-3} = -\frac{-x-y+z}{x-3} \text{ or } -\frac{z-x-y}{x-3}$$

When the numerator is in factored form, the signs of an even number of factors may be changed without affecting the sign of the numerator But when the signs of an odd number of factors in the numerator are changed, the sign of the numerator is changed The same, of course, is true of the denominator

Verify the fact that

$$\frac{(a-2)(b-3)(c-4)}{6} = \frac{(2-a)(3-b)(c-4)}{6}$$

You will notice that the signs of the first two factors of the numerator have been changed That is,

$$-(a-2) = -a+2 \text{ or } 2-a$$

and

$$-(b-3) = -b+3 \text{ or } 3-b$$

[A]

EXERCISES Supply the missing terms or factors

1 $\frac{2}{-5} = \frac{-2}{?} = -\frac{2}{?}$

2 $\frac{-a}{b} = \frac{a}{?}$

3 $\frac{a-b}{b-c} = -\frac{?}{b-c}$

4 $\frac{x^2-64}{2-x} = -\frac{?}{2-x}$

5 $\frac{x+y}{x-y-z} = -\frac{x+y}{?}$

6 $\frac{(a-b)(a+b)}{3} = -\frac{?\quad ?}{3}$

7 $\frac{(x-2)(x-4)}{-3} = \frac{?\quad ?}{3}$

8 $-\frac{(a-b)(a+b)}{4} = \frac{?\quad ?}{4}$

9 $\frac{(a+2)(3-a)}{(2-a)(a+3)(a-4)} = \frac{?\qquad ?}{(a-2)(a+3)(a-4)}$

Reducing Fractions to Lowest Terms[A]

A fraction is in lowest terms when its numerator and denominator have no common integral factor except 1 In algebra the word *term* has two meanings The terms of a fraction are the numerator and denominator of the fraction

To reduce a fraction to lowest terms, divide the numerator and denominator by their common factors

Example 1 Reduce $\frac{2\,a^2b}{8\,ab^2}$ to lowest terms

Solution 2, a, and b are factors common to both numerator and denominator Dividing both the numerator and denominator by $2\,ab$, we have

$$\frac{2\,a^2b}{8\,ab^2}=\frac{a}{4\,b}$$

Example 2 Reduce $\frac{a^2-4\,b^2}{a^2-ab-2\,b^2}$ to lowest terms

Solution When the numerator and denominator are polynomials, it is much easier to recognize the common factors and the resulting quotients if we first factor the numerator and denominator

$$\frac{a^2-4\,b^2}{a^2-ab-2\,b^2}=\frac{(a+2\,b)(a-2\,b)}{(a+b)(a-2\,b)}$$

Now we see that the numerator and denominator contain the common factor $a-2\,b$ Then dividing both numerator and denominator by $a-2\,b$, we have

$$\frac{(a+2\,b)\overset{1}{\cancel{(a-2\,b)}}}{(a+b)\underset{1}{\cancel{(a-2\,b)}}}=\frac{a+2\,b}{a+b}$$

The oblique lines in the fraction above are called cancellation marks

Cancellation marks are used in mathematics with two different meanings In the above example we divided both numerator and denominator by $a-2\,b$ In the numerator, 1 was placed above $a-2\,b$ to indicate that the quotient of $a-2\,b$ divided by $a-2\,b$ was 1 Also, 1 was placed below $a-2\,b$ in the denominator to indicate that the quotient of $a-2\,b$ divided by $a-2\,b$ was 1

In the expression $2\,a+b-2\,a$, oblique lines may be drawn through $2\,a$ and $-2\,a$ indicating that $2\,a-2\,a=0$ Thus $\cancel{2a}+b\cancel{-2a}=b$

Avoid drawing lines through identical expressions without thinking what operation you are performing and knowing the correct result

Example 3 Simplify $\dfrac{7x^2 - x^3 - 12x}{x^4 - 16x^2}$.

Solution The word *simplify* in mathematics means to put the given expression in simplest form. Here *simplify* means to reduce the fraction to lowest terms.

Arranging the numerator in order of descending powers of x, then changing the sign of the fraction and of the numerator,

$$\frac{7x^2 - x^3 - 12x}{x^4 - 16x^2} = \frac{-x^3 + 7x^2 - 12x}{x^4 - 16x^2} = -\frac{x^3 - 7x^2 + 12x}{x^4 - 16x^2}$$

Factoring,

$$-\frac{x^3 - 7x^2 + 12x}{x^4 - 16x^2} = -\frac{x(x^2 - 7x + 12)}{x^2(x^2 - 16)} = -\frac{x(x-3)(x-4)}{x^2(x+4)(x-4)}$$

Dividing both numerator and denominator by the common factors x and $x - 4$, we have

$$-\frac{\overset{1}{\cancel{x}}(x-3)\overset{1}{\cancel{(x-4)}}}{\underset{x}{\cancel{x^2}}(x+4)\underset{1}{\cancel{(x-4)}}} = -\frac{x-3}{x(x+4)} \quad \text{or} \quad \frac{3-x}{x(x+4)}$$

EXERCISES [A]

Simplify

1. $\dfrac{21}{91}$ $-\dfrac{7}{13}$ $\dfrac{3}{17}$
2. $\dfrac{9x^7y^{10}}{36x^{10}y^3}$
3. $\dfrac{a^2 + a}{a + 1}$
4. $\dfrac{m^2 - m - 12}{m^2 + 10m + 21}$
5. $\dfrac{x^3 + 27}{21 + x - 2x^2}$
6. $\dfrac{ab}{ax - ay}$
7. $\dfrac{am + an + bm + bn}{m^3 + n^3}$
8. $\dfrac{x^2 - 1}{1 - x^3}$
9. $\dfrac{2x^2 + 2x - 12}{6x^2 + 18x}$
10. $\dfrac{a^3 - ab^2}{a^2 - 2ab + b^2}$
11. $\dfrac{x^3 - 3x^2 + x - 3}{x^2 - 9}$
12. $\dfrac{3x^3 - 11x^2 + 10x}{25 - 9x^2}$
13. $\dfrac{a^2b^2 - 16b^2}{a^2b + 9ab + 20b}$
14. $\dfrac{a^3 + b^3}{a^4 + a^2b^2 + b^4}$
15. $\dfrac{xy - 3x - 2y + 6}{y^3 - 27}$
16. $\dfrac{a^2 - (b-2)^2}{4 - (a-b)^2}$

17. $\dfrac{x - x^4}{x^2 + x + 1}$

18. $\dfrac{a^3 + 2a^2 + a + 2}{a^3 - 5a^2 + a - 5}$

19. $\dfrac{ax}{a^2x^2 - ax}$

20 $\dfrac{15 + a - 2a^2}{a^3 - 27}$

21 $\dfrac{x^3 - y^3}{2x^2 - 2y^2}$

22 $\dfrac{xy^2 - 16x}{xy^2 + 9xy + 20x}$

(B)

23. $\dfrac{2 + a + 2a^2 + a^3}{a^3 - 5a^2 + a - 5}$

24. $\dfrac{3a^3 + 8a^2 + 5a}{3a^2 - a - 10}$

25 $\dfrac{a^4 - 16}{a^6 + 64}$

26 $\dfrac{2x^6 + 5x^3 - 3}{1 - 4x^6}$

27 $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{a^3 + 2a^2b + ab^2}$

28 $\dfrac{x^2 - 2xy + y^2 - z^2}{z^2 - x^2 + y^2 + 2yz}$

Multiplying Fractions (A)

The product of two or more fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators

If any or all the numerators or denominators are polynomials, we first try to factor them, and if there are factors common to both terms of the fraction, we divide both terms by the common factors The product of the remaining factors of the numerators forms the numerator of the resulting fraction, and the product of the remaining factors of the denominators forms the denominator of the resulting fraction

Example 1 $\dfrac{2a^2}{3b} \cdot \dfrac{15b^2}{8a}$

Solution $\dfrac{2a^2}{3b} \cdot \dfrac{15b^2}{8a} = \dfrac{30a^2b^2}{24ab} = \dfrac{5ab}{4}$

Example 2 $\dfrac{ab - b^2}{2a} \cdot \dfrac{2a + 2b}{a^2b - b^3}$

Solution $\dfrac{\cancel{b}\cancel{(a-b)}}{\cancel{2}a} \cdot \dfrac{\cancel{2}\cancel{(a+b)}}{\cancel{b}\cancel{(a+b)}\cancel{(a-b)}} = \dfrac{1}{a}$

EXERCISES

[A]

Multiply as indicated

1. $\frac{6a^2}{5b}\;\frac{45b^3}{18a^3}$

2. $\frac{7c^{2+n}}{11a^2}\;\frac{132a^2}{56c^{1+n}}$

3. $a^2b\left(\frac{3}{4ab^2}\right)$

4. $y\;\frac{x}{y}$

5. $\frac{m-n}{m^2-1}\;\frac{m-1}{m^2-n^2}$

6. $\frac{3}{a-b}\;\frac{(a-b)^2}{18}$

7. $\frac{b^3}{a^2-b^2}\;\frac{a+b}{4ab}$

8. $\frac{x^2-8x+15}{8}\cdot\frac{6}{x^2-10x+21}$

9. $\frac{y^2-y-6}{2y^2+8y+8}\cdot\frac{6}{36-4y^2}$

10. $\frac{(a+b)^2}{a^3+b^3}\cdot\frac{a^2-ab+b^2}{a^2+2ab+b^2}$

11. $\frac{x^2-2x-8}{x^4-x^3}\;\frac{x^3-4x^2+4x}{5x^2-30x+40}$

12. $\frac{a}{a-b}\;\frac{b-a}{a}$ (5x − 4)(x − 1

13. $\frac{x^3+1}{2x^2+4x}\;\frac{3x^2-3x+3}{(2x+4)}$

14. $\frac{27a^3+8}{6a^2+19a+10}\;\frac{4a^2-25}{9a^2-6a+4}$

15. $\frac{x^2-6x+5}{xy^2-y^2}\;\frac{xy}{35-2x-x^2}$

16. $\frac{a^6+b^6}{a^3-b^3}\;\frac{a-b}{a^2+b^2}$

17. $\frac{abx^2-aby^2}{x^2+2xy+y^2}\;\frac{xy-y^2}{x^2-2xy+y^2}$

18. $\frac{x^2y-16y^3}{x^2+3xy-4y^2}\;\frac{y-x}{xy^2-4y^3}$

[B]

19. $\frac{a^3-3a^2-a+3}{14ab^2}\;\frac{21a^2b}{a^3-2a^2-2a-3}$

20. $\frac{x^2-xy+y^2}{xy}\;\frac{x^2-y^2}{xy}\;\frac{2xy}{x^3+y^3}$

21. $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\;\frac{a+b+c}{c+b-a}$

Dividing Fractions [A]

To divide one fraction by another, multiply the dividend by the reciprocal of the divisor This means "invert the divisor and proceed

as in multiplication." The reciprocal of $\frac{2a}{5}$ is $\frac{5}{2a}$, and the reciprocal of a is $\frac{1}{a}$.

Example $\frac{a^3 - ab^2}{4} \div \frac{a^2 - 2ab + b^2}{8}$

Solution $$\frac{a(a+b)\cancel{(a-b)}}{\underset{1}{\cancel{4}}} \cdot \frac{\overset{2}{\cancel{8}}}{(a-b)\cancel{(a-b)}} = \frac{2a(a+b)}{a-b}$$

[A] **EXERCISES**

1 $\frac{6x^2y^3}{5a^3b} \div \frac{2xy}{15a^2b^2}$

2 $\frac{12m^{x+2}}{n^2} \div 6m^{x+2}$

3 $\frac{x+y}{y^{n+1}} \div \frac{x^2-y^2}{y^{n+3}}$

4 $\frac{m^2-mn}{n} \div \frac{m}{n}$

5 $\frac{b^2-2b+1}{6} \div (b^2-1)$

6 $\frac{a^6+b^6}{18a} \div \frac{a^2+b^2}{15a^2}$

7 $\frac{t}{t-3} \div \frac{t^4}{2}$

8 $\frac{a^3-8}{a^2-2a+4} \div (a^2-4)$

9 $\frac{y^2-x^2}{y^2} \div \frac{x+y}{2y}$

10 $\frac{m^3+8n^3}{14} \div \frac{m^2+4mn+4n^2}{21}$

11 $\frac{1-m}{m-3} \div \frac{m^4-1}{12m-36}$

12 $\frac{2t}{4t^2-1} \div \frac{6t^3}{6t+3}$

13 $\frac{4x^2+8x+3}{2x^2-5x+3} \div \frac{1-4x^2}{6x^2-9x}$

14 $\frac{(a-b)^3}{b^3+c^3} \div \frac{a^2-2ab+b^2}{b^2-bc+c^2}$

[B]

15 $\frac{x^2+y^2-z^2-2xy}{x^2+y^2-z^2+2xy} \div \frac{x-y-z}{(x+y-z)^2}$

16 $\frac{a^2-b^2}{a^2} \cdot \frac{a^2-ab+b^2}{a^2} \div \frac{a^3+b^3}{a^4}$ $4x^2+16x+16$

17 $\frac{m^2+2m+1}{m^2+4m} \cdot \frac{m^2-16}{m^2-3m-4} \div \frac{m^2-2m-3}{m^2-m}$

18 $\frac{a^2-4a-ab+4b}{2a^2-6a-ab+3b} \div \frac{2a^2-8a+ab-4b}{a^2-4a}$

Adding and Subtracting Fractions[A]

Fractions may be added or subtracted if they have a common denominator. In order to add or subtract fractions that do not have a

common denominator, it is first necessary to change the fractions to equivalent fractions having a common denominator. To keep the work as simple as possible we use the lowest common denominator (L.C.D.). The L.C.D. is the L.C.M. of the denominator (see page 86).

To find the lowest common denominator:

1. Factor each denominator.
2. Write the product of all the different factors, giving each factor the largest exponent of that factor in any denominator.

Rule for Adding (or Subtracting) Fractions

1. *Change them, if necessary, to equal fractions having the same denominator (usually the L.C.D.).*
2. *Place the sum (or difference) of the numerators over the common denominator.*
3. *Reduce the resulting fraction to lowest terms.*

Example 1. Combine $\frac{3x}{5} - \frac{x}{10} + \frac{x}{2}$.

$\frac{6x - x + 5x}{10}$

Solution. By inspection the L.C.D. is 10.

$\frac{6}{10} = \boxed{x}$

$$\frac{3x}{5} - \frac{x}{10} + \frac{x}{2}$$
$$= \frac{6x}{10} - \frac{x}{10} + \frac{5x}{10}$$
$$= \frac{10x}{10}$$
$$= x$$

Example 2. Simplify $\frac{a}{a^2 - 2ab + b^2} - \frac{b}{a^2 - b^2}$.

Solution. Factoring the denominators, we have

$$\frac{a}{a^2 - 2ab + b^2} - \frac{b}{a^2 - b^2} = \frac{a}{(a-b)^2} - \frac{b}{(a+b)(a-b)}$$

The different factors are $(a - b)$ and $(a + b)$. The highest exponent of $(a - b)$ is 2, and the highest exponent of $(a + b)$ is 1. Therefore the L.C.D. is $(a - b)^2(a + b)$. Changing each fraction to an equivalent fraction having $(a - b)^2(a + b)$ as the denominator, we have

$$\frac{a}{(a-b)^2} - \frac{b}{(a+b)(a-b)} = \frac{a(a+b)}{(a-b)^2(a+b)} - \frac{b(a-b)}{(a-b)^2(a+b)}$$
$$= \frac{a^2 + ab - ab + b^2}{(a-b)^2(a+b)} = \frac{a^2 + b^2}{(a-b)^2(a+b)}$$

In adding or subtracting fractions, the answer should be given in lowest terms. Study Example 3.

Example 3. Simplify $\frac{1}{6x}+\frac{1}{3x-6}-\frac{1}{2(x+2)}$

6x(x-2)(x+2)

Solution.

$$\frac{1}{6x}+\frac{1}{3(x-2)}-\frac{1}{2(x+2)}$$

3x(x-2)

$$=\frac{(x+2)(x-2)}{6x(x-2)(x+2)}+\frac{2x(x+2)}{6x(x-2)(x+2)}-\frac{3x(x-2)}{6x(x-2)(x+2)}$$

3-2-1

$$=\frac{x^2-4+2x^2+4x-3x^2+6x}{6x(x-2)(x+2)}=\frac{10x-4}{6x(x-2)(x+2)}$$

$$=\frac{5x-2}{3x(x-2)(x+2)}$$

-2x-1

x

(A) **ORAL EXERCISES**

Combine

1. $\frac{3}{4}+\frac{1}{2}$
2. $\frac{7}{8}-\frac{1}{2}$
3. $\frac{3}{5}+\frac{2}{5}$
4. $\frac{7}{9}-\frac{1}{9}$
5. $\frac{2}{c}-\frac{3}{c}$
6. $\frac{3}{x+1}-\frac{2}{x+1}$
7. $\frac{b}{b-c}-\frac{c}{b-c}$
8. $1+\frac{b}{c}$
9. $\frac{a}{b}-1$
10. $\frac{b}{c}+d$
11. $\frac{m}{n}+\frac{2m}{n}$
12. $\frac{2x}{x-1}-\frac{2}{x-1}$
13. $1-\frac{a}{b}$
14. $\frac{1}{a}+\frac{1}{b}$
15. $\frac{1}{c}-\frac{1}{d}$

8

6+4

(A) **EXERCISES**

Combine

1. $\frac{a}{2}+\frac{b}{4}+\frac{c}{6}$
2. $\frac{3}{2x^3}-\frac{2}{3x^2}+\frac{1}{x}$
3. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$
4. $\frac{4x-7}{8}-\frac{3x-5}{12}$
5. $\frac{2b-5}{12}+\frac{4b-1}{20}$
6. $\frac{x-5}{4x}-\frac{3x-1}{x}$
7. $\frac{2c}{3-2c}+1$
8. $\frac{5a+3b}{2a^2b}-\frac{3a+4b}{ab^2}$

9 $x - 5 - \dfrac{2 + 13x}{3x}$

10 $\dfrac{2 + 5x}{3x} - 2x + 3$

11 $\dfrac{x^2 - 5}{x^3 - 1} - \dfrac{x + 1}{x^2 + x + 1}$

12 $a + \dfrac{a - 2}{3 - a} + \dfrac{4 - a^2}{a^2 - 9}$

13 $1 + \dfrac{a}{1 - a}$

14 $\dfrac{x}{x^2 - 16} - \dfrac{x + 1}{x^2 - 5x + 4}$

15 $\dfrac{2s}{t^2 - s^2} - \dfrac{s}{t - s} + \dfrac{t}{t + s}$

16 $\dfrac{2}{b^2 - ab} + \dfrac{2}{a^2 - ab}$

17 $\dfrac{3}{12 + x - x^2} + \dfrac{2}{x^2 - 9}$

18 $\dfrac{5}{5x - x^2} - \dfrac{2}{x^2 - 25}$

19 $\dfrac{2x - 1}{x} + \dfrac{x + y}{x + y} - \dfrac{y + 1}{y}$

20 $\dfrac{1}{x + y} + \dfrac{1}{x - y} - \dfrac{2x}{x^2 - y^2}$

[B]

21 $\dfrac{t - 3}{t^2 - 3t + 2} - \dfrac{t - 2}{t^2 - 4t + 3} - \dfrac{t - 1}{t^2 - 5t + 6}$

22 $\dfrac{s - 1}{s^2 - 9s + 20} + \dfrac{4s - 5}{s^2 - 8s + 15} - \dfrac{3s - 6}{s^2 - 7s + 12}$

23 $\dfrac{5}{3 - 2x} + \dfrac{3}{2x - 3} - \dfrac{x - 3}{2x^2 - x - 3}$

24 $\dfrac{x + y}{x^2 + 2xy - 3y^2} - \dfrac{x - 2y}{x^2 - y^2} + \dfrac{2x + y}{x^2 + 4xy + 3y^2}$

25 $\dfrac{a}{1 - a^3} + \dfrac{1}{a^2 - 1} - \dfrac{2}{a^2 + a + 1}$

Mixed Expressions [A]

A mixed expression is one containing a fraction and one or more terms that are not fractions

$2\frac{1}{3}$, $\dfrac{1}{a} + a$, and $2x - 3 - \dfrac{5}{x + 4}$ are mixed expressions

In problems involving multiplication or division, mixed expressions should first be reduced to fractional form

Example Simplify $\left(\dfrac{1}{x} - 1\right)\left(x + 1 + \dfrac{1}{x}\right)$

Solution $\left(\dfrac{1}{x} - 1\right)\left(x + 1 + \dfrac{1}{x}\right) = \dfrac{1 - x}{x} \cdot \dfrac{x^2 + x + 1}{x} = \dfrac{1 - x^3}{x^2}$

$1 - x)(x^2 + x + 1)$

EXERCISES [A]

Simplify

1 $\left(x - \frac{1}{x}\right) \div \left(x + \frac{1}{x} - 2\right)$

2 $\left(\frac{1}{x} + \frac{1}{y}\right) \div \left(\frac{1}{x} - \frac{1}{y}\right)$

3 $\left(a + \frac{a}{b}\right) \div \left(1 + \frac{1}{b}\right)$

4 $\left(1 - \frac{2}{3a}\right)\left(\frac{9a}{9a^2 - 4}\right)$

5 $\left(m^3 + m - [illegible] - \frac{1}{m} + \frac{1}{m^2}\right)$

6 $\left(\frac{1}{x} - 1\right) \div \left(x - [illegible]\right)$

7 $\left(\frac{x}{y} - \frac{y}{x}\right)\left(x - \frac{x}{x + y}\right)$

8 $\left(1 + \frac{a + b}{a - b}\right)\left(1 + \frac{a - b}{a + b}\right)$

9 $\left(a - 5 + \frac{8}{a + 1}\right)\left(a + 2 + \frac{4}{a - 3}\right)$

10 $\left(\frac{t^2 - 1}{t^3 - 1} + \frac{t - 1}{t^2 + t + 1}\right) \div \left(\frac{t + 1}{t^2 - t + 1} + \frac{[illegible]}{t^3 + 1}\right)$ [B]

11 $\left(\frac{x}{y} + \frac{1}{x}\right)\left(x - \frac{x^3}{x^2 + y}\right)$

12 $\left(\frac{m}{n} - \frac{n}{m}\right)\left(1 + \frac{n}{m}\right) \div \left(\frac{n}{m} - \frac{m}{n}\right)$

13 $(a + 1)\left(\frac{1}{a^2 - 1} + 1 - \frac{1}{2(a - 1)}\right)$

Complex Fractions [A]

A complex fraction is one that has a fraction in either the numerator or the denominator, or both

$$\frac{\frac{x + 1}{x}}{5}, \quad \frac{a - b}{\frac{a}{b}} \quad \text{and} \quad \frac{1 + \frac{s}{t}}{1 - \frac{s^2}{t^2}}$$

are complex fractions The main fraction line is either made heavier or longer than the others to avoid confusion We can change [illegible] complex fractions to simple fractions (1) by multiplying both numerator and denominator of the complex fraction by the L C D of all the fractions in both the numerator and the denominator, or (2) by simplifying both numerator and denominator and then dividing the numerator by the denominator

Example Simplify $\dfrac{1 + \frac{1}{x}}{\frac{2}{x^2} - \frac{1}{2x}}$

Solution 1 The L C D of $\frac{1}{x}$, $\frac{2}{x^2}$, and $\frac{1}{2x}$ is $2x^2$

Then $$\frac{2x^2\left(1 + \frac{1}{x}\right)}{2x^2\left(\frac{2}{x^2} - \frac{1}{2x}\right)} = \frac{2x^2 + 2x}{4 - x}$$

Solution 2

$$1 + \frac{1}{x} = \frac{x}{x} + \frac{1}{x} = \frac{x+1}{x}$$

$$\frac{2}{x^2} - \frac{1}{2x} = \frac{4}{2x^2} - \frac{x}{2x^2} = \frac{4-x}{2x^2}$$

$$\frac{x+1}{x} \div \frac{4-x}{2x^2} = \frac{x+1}{\underset{1}{\cancel{x}}} \times \frac{\overset{2x}{\cancel{2x^2}}}{4-x}$$

$$= \frac{2x^2 + 2x}{4 - x}$$

EXERCISES

[A]

Simplify the following complex fractions Use either the method of solution 1 or the method of solution 2, as directed by your teacher

The numerical exercises in Exs 1-8 are of the type that occur in checking fractional equations

$\frac{21}{9-2} = \frac{21}{7} = 3$

1 $\dfrac{\frac{1}{2}+1}{\frac{1}{2}-1}$

2 $\dfrac{3}{1+\frac{1}{3}}$

3 $\dfrac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}$

4 $\dfrac{9-\frac{1}{4}}{3+\frac{1}{2}}$

5 $\dfrac{7}{3-\frac{2}{3}}$

6 $\dfrac{-\frac{7}{3}-1}{3}$

7 $\dfrac{1-\frac{3}{5}}{1+\frac{1}{5}}$

8 $\dfrac{\frac{7}{12}-\frac{1}{3}}{\frac{2}{3}-\frac{1}{2}}$

9 $\dfrac{\frac{1}{x}+y}{\frac{1}{x}-y}$

10 $\dfrac{a-\frac{1}{b}}{b-\frac{1}{a}}$

11 $\dfrac{1+\frac{a}{b}}{1-\frac{a^2}{b^2}}$

12 $\dfrac{1-\frac{x}{y}}{\frac{x-y}{y}}$

13 $\dfrac{\frac{a^2b}{c^2}}{\frac{ab^2}{c^2}}$

14 $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$

15 $\dfrac{a+1}{1+\frac{1}{a}}$

16 $\dfrac{a+\frac{a}{b}}{1+\frac{1}{b}}$

17 $\dfrac{a-1}{a-\frac{1}{a}}$

18 $\dfrac{x^2-\frac{1}{x}}{x^3-\frac{1}{x^3}}$

19 $\dfrac{\frac{x^2}{6}+\frac{5x}{12}-1}{\frac{x^2}{4}+\frac{11x}{12}-\frac{1}{3}}$

20 $\dfrac{x-3+\frac{2}{x}}{x-4+\frac{3}{x}}$

21 $\dfrac{\frac{x^3+27}{x^2-9}}{\frac{x^2-3x+9}{x+3}}$

22 $\dfrac{\frac{1}{x-y}-\frac{1}{x+y}}{\frac{2}{x^2-y^2}}$

23 $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a+b}{a-b}}$

[B]

Simplify

24 $\dfrac{\frac{x}{b^2}+\frac{a^4b^2}{x^3}-\frac{23a^2}{x}}{\frac{a^2b}{x^2}+\frac{5a}{x}+\frac{1}{b}}$

25 $\dfrac{\frac{x}{1+x}-\frac{1-x}{x}}{\frac{x}{1+x}+\frac{1-x}{x}}$

26 $\dfrac{\frac{a^2}{a^2b^2}-1}{\frac{ab}{a+b}-b}$

27 $\dfrac{\frac{x^2-3y}{a^2+1}}{\frac{9y^2-6xy+x^2}{1+a}}$

28 $\dfrac{\frac{3a+b}{a-b}-3}{1-\frac{a-3b}{a+b}}$

29 $\dfrac{\frac{x-1}{x+1}-\frac{x+1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$

Fractional Equations [A]

A fractional equation is one in which some function of the unknown appears in a denominator Thus $\frac{x}{x+2}+\frac{1}{x}=1$ is a fractional equation $\frac{x}{2}+\frac{3x}{4}=10$ is not a fractional equation because it does not have x in the denominator

In solving fractional equations we multiply both members of the equation by the L C D in order to clear the equation of fractions

In solving fractional equations each fraction should first be reduced to lowest terms

Multiplying both members of an equation by the same expression containing the unknown may introduce solutions of the new equation that are not solutions of the original equation

Equivalent Equations [A]

Two equations are equivalent when they have the same solutions

The four equations given at the right are equivalent equations, each having the root -2

$$\begin{aligned} 3(x-2)-(x-4) &= -6 \\ 3x-6-x+4 &= -6 \\ 3x-x &= 6-4-6 \\ 2x &= -4 \end{aligned}$$

Each of the following operations on equations leads to equivalent equations and may be used in the process of a solution

1 Addition of the same number to both members

2 Subtraction of the same number from both members

3 Multiplication (or division) of both members by the same number, provided that this number is not zero and does not involve the unknowns

When one equation is changed into an equivalent equation, no root is introduced and no root is lost

Defective Equations [A]

Now let us see how a root is lost in a solution

Consider the equation $x^2-4x=0$ Factoring the left member, we have $x(x-4)=0$ Dividing both members by x, we get $x-4=0$ The roots of the original equation are 0 and 4, but only 4 is a root of $x-4=0$ The root 0 was lost when we divided by x

Again, consider the equation $x^2-3x+2=x-1$ Factoring the left member, we get $(x-1)(x-2)=x-1$ Dividing both members by $x-1$, we get $x-2=1$ and $x=3$ The roots of $x^2-3x+2=x-1$ are 1 and 3, but $x-2=1$ has only the root 3 When we divided by $x-1$, the root 1 was lost

With respect to the equation $x^2-4x=0$, the equation $x-4=0$ is *defective*, and with respect to the equation $x^2-3x+2=x-1$,

the equation $x - 2 = 1$ is *defective* When a derived equation lacks one or more roots of the original equation, it is defective

Dividing both members of an equation by the same rational integral function of the unknown leads to a defective equation

Redundant Equations [A]

Sometimes derived equations have roots which are not roots of the original equations Let us see how such a root is introduced

Consider the equation $3x - 1 = x + 3$, whose root is 2 Multiplying both members of the equation by $x - 1$ we get $3x^2 - 4x + 1 = x^2 + 2x - 3$ This equation has the roots 2 and 1 With respect to the equation $3x - 1 = x + 3$, it is *redundant*

A derived equation is redundant if it contains all the roots of the original equation and one or more additional ones The derived equation $3x^2 - 4x + 1 = x^2 + 2x - 3$ was obtained by multiplying both members of the original equation by $x - 1$, a function of x

Multiplying both members of an equation by the same integral function of the unknown leads to a redundant equation.

Redundant equations often occur in solving fractional equations and radical equations In such cases not all the roots of the derived equations are roots of the original equations

A number which is a root of a derived equation but not of the original equation is sometimes called an *extraneous root*

Example 1 Solve $\dfrac{x}{x-2} + \dfrac{2}{x+2} = \dfrac{x^2+4}{x^2-4}$. $x^2 \quad 2x-4$

Solution Multiplying both members of the equation by the L C D, $(x-2)(x+2)$, gives

$$\frac{x(x+2)(x-2)}{x-2} + \frac{2(x-2)(x+2)}{x+2} = \frac{(x^2+4)(x+2)(x-2)}{(x+2)(x-2)}$$

$$x^2 + 2x + 2x - 4 = x^2 + 4$$

$$4x = 8$$

$$x = 2$$

Substituting this value of x in the original equation, we get

$$\frac{2}{2-2} + \frac{2}{2+2} = \frac{4+4}{4-4}, \quad \text{or} \quad \frac{2}{0} + \frac{2}{4} = \frac{8}{0}$$

(cont on p 116)

Since division by zero is an excluded operation, 2 is not a solution of the original equation. The original equation has no solution. The apparent solution $x = 2$ was introduced when we multiplied both members of the equation by $(x+2)(x-2)$.

The example above helps to explain why no number can be called a root of an equation until it has been proved a root.

Example 2 Solve $\dfrac{2x}{x+1} = 2 - \dfrac{5}{2x}$.

Solution. The L.C.D. is $2x(x+1)$.

Multiplying both members by the L.C.D., we have

$$\frac{2x \cdot 2x(x+1)}{x+1} = 2 \cdot 2x(x+1) - \frac{5 \cdot 2x(x+1)}{2x}$$

$$4x^2 = 4x^2 + 4x - 5x - 5$$

$$x = -5$$

PROOF $\qquad \dfrac{2x}{x+1} = 2 - \dfrac{5}{2x}$

Does $\qquad \dfrac{-10}{-4} = 2 - \dfrac{5}{-10}?$

Does $\qquad 2\frac{1}{2} = 2\frac{1}{2}?$ Yes.

Therefore $x = -5$ is a solution of the equation.

Example 3 Solve $\dfrac{3x}{x^2-9} - \dfrac{6-x}{x^2+3x} = \dfrac{4}{x+3}$.

Solution. $\qquad \dfrac{3x}{(x+3)(x-3)} - \dfrac{6-x}{x(x+3)} = \dfrac{4}{x+3}$

The L.C.D. is $x(x+3)(x-3)$.

$$\frac{3x \cdot x(x+3)(x-3)}{(x+3)(x-3)} - \frac{(6-x)x(x+3)(x-3)}{x(x+3)} = \frac{4x(x+3)(x-3)}{x+3}$$

$$3x^2 + x^2 - 9x + 18 = 4x^2 - 12x$$

$$3x = -18$$

$$x = -6$$

PROOF Does $\dfrac{-18}{36-9} - \dfrac{6+6}{36-18} = \dfrac{4}{-6+3}?$

Does $\qquad -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}?$

Does $\qquad -\frac{4}{3} = -\frac{4}{3}?$ Yes.

Therefore $x = -6$ is a solution of the equation.

EXERCISES

[A]

Solve

1 $\frac{5}{x} = \frac{6}{x} - \frac{1}{2}$

2 $\frac{7}{2x} + \frac{1}{3} = \frac{1}{x} + \frac{7}{6}$

3 $\frac{x-3}{x} = \frac{1}{x} + \frac{4}{3}$

4 $\frac{15}{x} - \frac{2x-5}{4x} = \frac{11}{4}$

5 $\frac{x}{x+1} + \frac{3x-2}{x} = 4$

6 $\frac{2}{y+1} = \frac{1}{y-2}$

7. $\frac{1}{y} - \frac{8}{y^2} = \frac{1}{y^2} + \frac{1}{3y}$

8 $\frac{x-2}{x-3} = \frac{x-1}{x+1}$

9 $\frac{3}{y+1} - \frac{5}{y+3} = 0$

10 $\frac{2x-3}{3x+2} = \frac{2x+1}{3x-2}$

11 $\frac{3}{x} - \frac{1}{x-2} = \frac{2}{x+2}$

12 $\frac{6}{3x-1} + \frac{1}{2} = \frac{14}{6x-2}$

13 $\frac{1}{x^2+8} = \frac{1}{x^2+4x+4}$

14 $\frac{y}{y-1} + \frac{2(y-1)}{y} = 3$

15 $\frac{6t-2}{2t-1} = \frac{9t}{3t+1}$

16 $\frac{1}{y} + \frac{2}{y+1} = \frac{3}{y+2}$

17 $\frac{4}{x-3} + \frac{2x}{x^2-9} = \frac{1}{x+3}$

18 $\frac{6x^2-2x+5}{3x^2-2} = 2$

*19 $\frac{3x}{x^2} - \frac{5}{x+6} = \frac{1}{x^2+6x}$

20 $\frac{x}{x+2} - \frac{x}{x-2} = \frac{x+20}{x^2-4}$

[B]

Solve

21 $\frac{4x+3}{x^2-x-6} + \frac{x-4}{x+2} = \frac{x}{x-3}$

22 $\frac{x^2+1}{x^2-1} - \frac{2}{x+1} = \frac{x}{x-1}$

23. $\frac{3x-12}{4x^2-9} - \frac{3x}{4x^2-16x+15} = 0$

24 $\frac{x^2}{x^3-1} + \frac{x}{x^2+x+1} - \frac{2}{x-1} = 0$

25 $\frac{x}{x-3} - \frac{2x+1}{x-4} + \frac{x^2}{x^2-7x+12} = 0$

*All fractions in equations should be reduced to lowest terms before clearing of fractions

EXERCISES [A]

Solve the following literal equations for the letters indicated

1 $\frac{x}{a} = b$ for x 3 $\frac{y}{c} = 5$ for y 5 $\frac{3}{5} = \frac{a}{x}$ for x

2 $\frac{a}{x} = b$ for x 4 $\frac{3}{y} = c$ for y 6 $\frac{a}{3} = \frac{6}{x}$ for x

7 $\frac{1}{y} - \frac{1}{b} = \frac{1}{a} - \frac{1}{y}$ for y

8 $\frac{b}{y} = \frac{a}{y} - c(a - b)$ for y

9 $F = \frac{wa}{g}$ for g

10 $a = \frac{v - v_0}{t}$ for v_0

11 $V = \frac{Q}{r_1} - \frac{Q}{r_2}$ for Q

12 $\frac{E}{e} = \frac{R + r}{r}$ for e

13 $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ for R

14 $\frac{1}{P} = \frac{1}{f_1} + \frac{1}{f_2}$ for P

15 $S = \frac{a - ar^n}{1 - r}$ for a

16 $E = RI + \frac{rI}{n}$ for I

Miscellaneous Problems [A]

Example 1 Using a tractor, Frank can plow a field in 18 hours If Frank uses the tractor and his father uses horses to pull the plows, working together they can plow the field in 15 hours How long will it take the father using horses to plow the field?

Solution

Let $x =$ the number of hours for the father using horses to plow the field

$\frac{15}{18} =$ the part of the field Frank can plow in 15 hours

$\frac{15}{x} =$ the part of the field the father can plow in 15 hours

Since they can plow the *whole* field in 15 hours

$$\frac{15}{18} + \frac{15}{x} = 1$$

M_{18x} $15x + 270 = 18x$

Solving $x = 90$

The father can plow the field in 90 hours

PROOF In 15 hours Frank can plow $\frac{15}{18}$ or $\frac{5}{6}$ of the field and his father can plow $\frac{15}{90}$, or $\frac{1}{6}$ of the field $\frac{1}{6} + \frac{5}{6} = \frac{6}{6}$

Example 2 The denominator of a certain fraction is 5 more than the numerator If 3 is added to both terms of the fraction, the resulting fraction equals $\frac{1}{2}$ Find the fraction

Solution Let $x =$ the numerator of the fraction
Then $x + 5 =$ the denominator of the fraction
and $\frac{x}{x+5}$ represents the fraction

Then
$$\frac{x+3}{x+8} = \frac{1}{2}$$
$$2x + 6 = x + 8$$
$$x = 2$$

The fraction is $\frac{2}{7}$

PROOF
$$\frac{2+3}{7+3} = \frac{5}{10} = \frac{1}{2}$$

(A)

1 The numerator of a certain fraction is 4 less than the denominator If both numerator and denominator are increased by 6, the resulting fraction is $\frac{2}{3}$ Find the fraction

2. The denominator of a certain fraction is 3 less than twice the numerator If the numerator is increased by 8 and the denominator by 35, the resulting fraction is $\frac{1}{3}$ Find the fraction

3 Two men can do a piece of work in 8 hours If one man can do the work alone in 14 hours, how long would it take the other man alone to do the work?

4 Ann can do a piece of work in 8 hours and Barbara can do the work in 10 hours How long will it take them together to do the work? $\frac{x}{8} + \frac{x}{10} =$

5. *John and Harry working together can paint a house in 10 days* When working alone, John can paint the house in 30 days How long will it take Harry working alone to paint the house?

6. If 12 is subtracted from both terms of a certain fraction, the value of the fraction becomes $\frac{2}{11}$ If 5 is added to the numerator the value of the fraction becomes $\frac{1}{2}$ Find the *fraction*

7. A man works for 12 days on a job and is then joined by another man, they finish the work in 8 more days The second

man could have done the job alone in 40 days How long would it have taken the first man to do the work?

8 Fred averaged 48 m p h when driving from his home to the Rose Bowl and 40 m p h when returning home on the same route Find the distance along this route from his home to the Rose Bowl if his time returning was 1 hour 4 minutes longer than his time going

9 A pilot has safe flying time of 88 minutes How far east can he fly when there is no wind if his air speed going is 300 m p h and his air speed returning is 250 m p h?

10 How far west can a pilot go and return in 3 hours if his speed in still air is 450 m p h when the wind is blowing from the west 50 m p h?

11 One boy can mow a lawn in 1 hour He and his brother working together can mow the lawn in 20 minutes How long would it take the brother to mow the lawn alone?

12 Two pumps when working together can fill a swimming pool in 24 hours When operated alone one pump requires 20 more hours than the other to fill the pool How long will it take each pump to fill the pool?

13 A tank can be filled by an inlet pipe in 4 hours and emptied in 6 hours by a drain pipe How long will it take to fill the tank if both pipes are open?

14 A tank can be filled by an inlet pipe in 8 hours when the drain pipe is open and in 5 hours if the drain pipe is closed How long will it take to empty a full tank when the inlet pipe is closed?

[B]

15 A can do a piece of work in 5 days B in 4 days and C in 3 days How many days will it take them to do it working together?

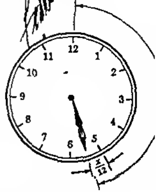
16 Albert, Bob and Charles can do a piece of work in 3 days If Albert can do the same piece of work in 8 days, and Bob in 10 days, how many days will it take Charles to do the work?

17 A cistern can be filled by 2 pipes in 4 hours and 5 hours, respectively, and can be emptied by a third pipe in 6 hours

How many hours will it take to fill the cistern if all the pipes are running?

18 How many minutes after 5 o'clock will the hands of a clock first be together?

Suggestions Let x = the number of minute-distances that the minute hand goes Then $\frac{x}{12}$ = the number of minute-distances that the hour hand goes. From the figure, $x - \frac{x}{12} = 25$



19. How soon after 7 o'clock will the hour and minute hands of a clock first be together?

20 How soon after 8 o'clock will the hands of a clock first be opposite each other?

21. *How soon after 3 o'clock will the hands of a clock first be 5 minute-distances apart?*

22 If 260 is divided by a certain number, the quotient is 15 and the remainder is 5 What is the number?

23 One automobile travels 10 miles an hour faster than a second automobile The first automobile travels 320 miles in the same time that the second automobile travels 240 miles Find the rate of each automobile

24. The sum of the numerator and denominator of a certain fraction is 104 The value of the fraction is $\frac{9}{17}$ Find the fraction

25. Frank can walk a mile in 3 minutes less time than his sister and he can walk 5 miles while his sister walks 4 miles Find the rate of each

26 When working together Anne, Bernice, and Christine can do a piece of work in 1 hour and 20 minutes To do the work alone, Christine would need twice as much time as Anne and 2 hours more than Bernice How long would it take each girl working alone to complete the work?

27. Two boys, one starting at one end of a cinder track and the other starting at the other end of the track, run toward

each other and meet in 12 seconds If it takes the slower boy 10 seconds longer than the faster boy to run the length of the track, and the rate of the faster boy is 9 yards per second, how long will it take each boy to run the length of the track?

Checking Your Understanding of Chapter 4

You should now determine if you have mastered Chapter 4 Be sure that you know

Do you need to review?

	PAGE
1 The fundamental principle of fractions	101
2 How to reduce fractions to lowest terms	102
3 How to add, subtract, multiply, and divide fractions	105–108
4 How to simplify complex fractions	111
5 How to solve fractional equations	113
6 How to solve verbal problems	118
7 How to spell and use the following expressions	

MATHEMATICAL VOCABULARY

	PAGE		PAGE
complex fraction	111	numerator	102
defective equation	114	reciprocal	107
fractional equation	113	redundant equation	115

CHAPTER REVIEW

[A]

Simplify

1. $\frac{8ab^2x}{4a^2b-8ab^2}$

2 $\frac{27a^3-1}{9a^2-6a+1}$

3 $\frac{6x^2-5x-6}{6x^2}\cdot\frac{9x^3}{9x^2+27x+14}$

4 $\frac{x^2-5}{x^3-1}-\frac{x+1}{x^2+x+1}$

5 $\frac{4x^2-1}{6x^2-9x}\div\frac{4x^2+8x+3}{2x^2-5x+3}$

6 $(m^2+m+1)\left(1-\frac{1}{m}+\frac{1}{m^2}\right)$

7 $a+\dfrac{2}{3-a}+3+\dfrac{4}{a+3}$

8. $\dfrac{a+b}{a-b}+\dfrac{1}{a+b}-\dfrac{a^2+b^2}{a^2-b^2}$

9. $\left(1+\dfrac{2y^3}{x^3-y^3}\right)\left(\dfrac{x-y}{x+y}\right)$

10 $\dfrac{x+\dfrac{27}{x^2}}{x-3+\dfrac{9}{x}}$

11. Solve $\dfrac{y}{y-3}+\dfrac{y^2}{y^2-7y+12}=\dfrac{2y+1}{y-4}$

12 Solve $S=\dfrac{rl-a}{r-l}$ for l

13 One pipe alone will fill a tank in 10 hours A second pipe alone will fill it in $7\frac{1}{2}$ hours If the first pipe were open for 8 hours and then closed, how long would it take the second pipe to finish filling the tank?

(II)

Simplify

14 $\dfrac{a-b+a^2-b^2}{a^2+2ab+b^2-1}$

15 $\dfrac{y^4-x^2y^2-y^2+x^2}{(y^2-2y+1)(x^2-y^2)}$

16 $\dfrac{ax-ay+bx-by}{a^2+2ab+b^2+a+b}\div\dfrac{ax-ay-bx+by}{a^2+ab+a}$

17 $\left(y-1-\dfrac{5}{y+3}\right)\div\left(3-\dfrac{2y+2}{y+2}\right)$

18 $\dfrac{\dfrac{x+y}{x}-\dfrac{x-y}{y}-\dfrac{2x}{y}}{\dfrac{2x-y}{y}-\dfrac{y}{x}}$

19. Solve for x $\dfrac{x+a}{a}-\dfrac{x+b}{b}=2(a^2-b^2)$

20 Solve $\dfrac{3}{1-3y}+\dfrac{9-2y}{3y^2-7y+2}+\dfrac{5}{2-y}=0$

21 The denominator of a certain fraction is 5 more than twice the numerator If 3 is added to both terms of the fraction, it equals $\frac{3}{7}$ Find the fraction

22. The quotient of seven times a certain number and a number which is 3 more than the first number is 4 Find the number

CHAPTER TESTS

[Test A]

Simplify

1 $\dfrac{x+x^4}{x^2-x+1}$

2 $\dfrac{ab^3}{a^2-ab} \div \dfrac{(a-b)^2}{ab}$

3 $\dfrac{2x^2+5x-3}{4x^2-1} \div \dfrac{x^3+27}{2x^2+x}$

4 $\dfrac{3}{x^2-xy} - \dfrac{3}{y^2-xy}$

5 $\dfrac{1}{x+y} + \dfrac{2x}{x^2-xy+y^2}$

6 $\left(a-3+\dfrac{4}{a+1}\right)\left(a+3+\dfrac{4}{a-1}\right)$

7 $\dfrac{1+\dfrac{x}{y}}{1-\dfrac{x^2}{y^2}}$

8 Solve for R $\quad \dfrac{E}{e} = \dfrac{R+r}{r}$

9 Solve $\dfrac{2x}{x+6} - \dfrac{x^2+6}{x^2+6x} = \dfrac{x+4}{x}$

10 Mr Adams and his son working together can paint a house in 3 days Mr Adams working alone can paint it in 5 days How long will it take the son to paint the house?

[Test B]

Simplify

1 $\dfrac{3a^2}{a^2+a+1} - \dfrac{2a}{a-1} - \dfrac{a^3}{a^3-1}$

2 $\dfrac{2x^2-7x-15}{3x^2-8x-3} \cdot \dfrac{9x^2-1}{4x^2-9} \div \dfrac{x^2+3x-10}{2x^2-9x+9}$

3 $\left(x+1-\dfrac{x}{x-1}\right)\left(x-1+\dfrac{x}{x+1}\right)$

4 $\dfrac{1-\dfrac{2a-1}{a-1}}{1-\dfrac{1}{a+1}}$

5 Solve $\dfrac{x}{x-4} - \dfrac{2x-1}{x+3} + \dfrac{x^2}{x^2-x-12} = 0$

6 Solve for n $\quad F = RI + \dfrac{rI}{n}$

7 The denominator of a certain fraction is 5 less than the numerator If the numerator is increased by 9, the fraction is equal to $\frac{3}{2}$ Find the fraction

8. The average speed of one airplane is 200 miles an hour faster than that of a certain train If the train starts at 2 A M and the airplane starts from the same place at 10 A M, they will have traveled the same number of miles by noon Find the rate of the airplane

CHAPTER 5

Functional Relations, Graphs, and Linear Functions

In this chapter you will study related changes

Changing Quantities[A]

Have you ever observed any object or quantity that does not change? At first you may answer in the affirmative, but, after careful thinking, you will agree that almost everything changes We can truthfully say that we live in an ever changing world

A change in one quantity is usually accompanied by a change in some other quantity For example, a change in the price of grain is usually accompanied by a change in the price of meat Also, an increase in the amount of employment in a community is often the cause of an increase in the population of the community

Scientists are constantly studying changes, endeavoring to discover the laws which show the relations of one kind of quantity to another

Many of these relations can be expressed mathematically Such relations are called functional relations As an example, let us consider one of the three laws of motion discovered by Sir Isaac Newton (1642–1727), the great English physicist The second of these laws, which expresses the relation between the change of velocity of a body and the force that causes the change, can be stated mathematically by the formula

$$\frac{F_1}{F_2} = \frac{a_1}{a_2},$$

F_1 and F_2 representing the forces and a_1 and a_2 representing the changes in velocity

Newton's second law of motion may be stated "Force is that which produces acceleration, and is jointly proportional to the mass, m, of the body and the acceleration, a, of the body" The formula $\frac{F_1}{F_2} = \frac{a_1}{a_2}$ applies to bodies having the same mass

Ways of Expressing Functional Relations[A]

There are four ways of expressing functional relations These relations may be shown by statements or rules, by tables, by graphs, and by formulas For example, suppose that we wish to show the relation between the Fahrenheit and centigrade thermometer readings We can express this relation by the following four methods

1. The statement

To find the centigrade temperature when the Fahrenheit temperature is known, subtract 32 from the Fahrenheit reading and multiply the remainder by $\frac{5}{9}$

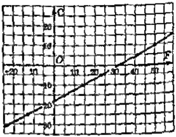
2 The table

The same relationship can be expressed for a limited number of cases by the following table

Fahrenheit	−13	−4	5	14	23	41	50
Centigrade	−25	−20	−15	−10	−5	5	10

3 The graph

The graph expresses the relationship between the two scales better than the table above. It shows the relation between the quantities for all values within its limits and pictures the relationship. The temperature on either scale can be found immediately when the temperature on the other scale is known.



4 The formula

The formula $C = \frac{5}{9}(F - 32)$ gives us the fourth method of showing functional relations. It expresses the rule by using algebraic symbols.

Each of these four forms of expressing functional relations has features that are not properties of the others. In most cases any one of these forms can be used to show a relationship. In this chapter you will learn how to substitute one form for another.

Making Formulas from Statements (A)

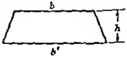
When a rule is changed into a formula, words are usually represented by letters that identify the words. For example, interest is usually represented by i, principal by p, rate by r, and time by t.

EXERCISES (A)

Make formulas for the following statements.

1. The area of a regular hexagon equals one half the product of its perimeter and the radius of its inscribed circle.

2. The area of a trapezoid is equal to one half the sum of its bases, b and b', multiplied by its altitude, h.



3 The cost of an article equals the selling price decreased by the margin

4. The selling price of an article is equal to its cost plus the profit plus the overhead

5 To find the volume of a cylinder, multiply its altitude by the area of its base

6 The volume of a pyramid equals one third the product of its base and altitude

7 To find the amount, add the principal and interest

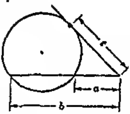
8 To find the diameter of a circle divide the circumference by π

9 The sum of the angles A, B, and C of a triangle is 180°

10 To find the number of degrees in the sum of the angles of a polygon subtract 2 from the number of its sides and multiply the difference by 180

11 The average of four numbers is found by dividing their sum by 4

12 The square of the hypotenuse of a right triangle equals the sum of the squares of the two legs



13 The square of a tangent from a point to a circle is equal to the product of a secant from the point and the external segment of the secant

14 The number of watt hours of electricity used is equal to the number of watts multiplied by the number of hours

15 To find the number of lines that are determined by a given number of points, no three of which are collinear, subtract 1 from the number of points, multiply the remainder by the number of points, and take half the product

Making Formulas from Data [A]

In the exercises above you made the formulas from given relationships In the following exercises you are to determine the relationships and make the formulas

[A] **EXERCISES**

Write a formula showing the relationship in each exercise

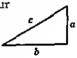
1 The number of seats in a room having 8 seats in a row

2 The cost of n articles at 10 cents each

3 The distance when the rate and time are given

4 The number of gallons that will flow through a pipe in h hours if 18 gallons will flow through it in 1 hour

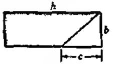
5 The hypotenuse c of a right triangle whose legs are a and b

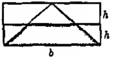


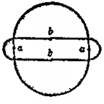
6 The cost of sending a package weighing more than a pound if the first pound costs 18 cents and each additional pound costs 1 45 cents

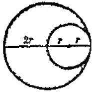
7 The cost of a taxi fare if the first mile costs 25 cents and each additional mile costs 10 cents

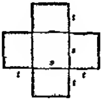
Write in simplest form the formula for the colored area in each exercise

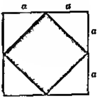
8 

9 

10 

11 

12 

13 

Kinds of Formulas [A]

Any formula is either mathematical or empirical A formula which expresses a general law, such as the law of falling bodies ($s = \frac{1}{2} gt^2$), or which is based upon mathematical calculations, is mathematical

A formula that is derived from experiments is empirical The word *empirical* means "depending entirely upon observation without due regard to science or theory" For example, the formula $p = \frac{a}{v}$, which states the relation between the pressure and volume of a confined gas, is empirical because it was developed from experiments

Constants and Variables [A]

The formula $C = \frac{22}{7} d$ tells how to find the approximate value of the circumference of a circle when the value of the diameter is known The table below contains six pairs of values of C and d that satisfy the formula There is an infinite number of pairs of such values

If d =	7	14	21	35	42	49
then C =	22	44	66	110	132	154

For each value of d there is a corresponding value of C When d is doubled, C is doubled, and when d is multiplied by a number, C is multiplied by that number Since C and d can have different values in the formula, they are called variables The number $\frac{22}{7}$ does not change in value It is called a constant The *subject* of the formula is C, and "$= \frac{22}{7} d$" is the *predicate*

The formula $C = \frac{22}{7} d$ tells what must be done to d to find C The value of C depends upon the value of d We call d the *independent variable* and C the *dependent variable* Some formulas contain more than one independent variable For example, $i = prt$ contains three independent variables, p, r, and t

An independent variable in a formula is a variable which is free to change A dependent variable in a formula is the variable whose change is caused by the change of the independent variable A constant *is a number whose value does not change during a discussion*

[A] **ORAL EXERCISES**

Upon what does each of the following depend?

1. The area of a rectangle, the area of a trapezoid whose bases are 10 inches and 14 inches
2. The circumference of a circle, the area of a circle
3. The cost of 5 pounds of meat
4. The time required for 10 men to do a piece of work

5 The time needed to travel 120 miles

6 The interest on \$100 for 1 year

7 The interest on \$200 at 4%

8 The amount of fence needed to enclose a rectangular lot 50 feet wide

9 The number of gallons of paint needed to cover a dwelling if one gallon covers 200 square feet

10 The volume of a rectangular solid having a given height

In the following formulas name the constants, the independent variables, and the dependent variables

11 $F = \frac{9}{5}C + 32$	16 $A = p + prt$
12 $C = S - M$	17 $i = 100\,rt$
13 $A = \pi r^2$	18 $A = p + i$
14 $C = 2\,\pi r$	19 $s = 16\,t^2$
15 $V = lwh$	20 $C = NP$

What the Word "function" Means [A]

The word "function" has several meanings Sometimes it means "duty", sometimes it means "social gathering", sometimes it means "use", but in mathematics it is used to indicate "dependence"

For each value of x in the equation $y = x^2$ there is one value of x^2 and one value of y, which equals x^2 Then the value of y (and x^2) depends upon the value of x We say that y (and x^2) is a function of x In the formula $F = \frac{9}{5}C + 32$, the value of $(\frac{9}{5}C + 32)$ depends upon the value of C Then $\frac{9}{5}C + 32$ is a function of C Why is F a function of C? In the formula $A = \pi r^2$ both A and πr^2 are the same function of r

If two variables are so related that for any value of one there is a value (or values) of the other, then the second variable is a function of the first variable

As you know, the words "function of x" can be represented by the symbol $f(x)$ Instead of saying "function of x," we can say "f of x" Thus, instead of writing $y = 2x^2 - 3x$, we may write $f(x) = 2x^2 - 3x$ and say that "f of $x = 2x^2 - 3x$" You should remember that in this case $f(x)$, y, and $2x^2 - 3x$ are the same function of x Again, if $s = \frac{1}{2}gt^2$, then $s = f(t) = \frac{1}{2}gt^2$

You should understand the difference between a function and an equation The expression $y = 3x + 2$ is an equation, while each of its two members are functions of x

If an algebraic expression contains more than one independent variable, the expression is a function of all the independent variables Thus in the formula $A = bh$, the expression bh (or A, which equals bh) is a function of both b and h Then $A = f(b, h) = bh$

If a function increases when the independent variable increases, it is an increasing function, if it decreases when the independent variable increases, it is a decreasing function

Evaluating a Function[A]

Evaluating a function is the process of finding the value of the function for some particular value of each letter

Example 1 Find the value of the function $4x - 1$ when $x = 3$

Solution Substitute 3 for x in $4x - 1$ If $x = 3$, then $4x - 1 = 4(3) - 1 = 12 - 1 = 11$

Example 2 Evaluate $f(x) = x^2 - 5x + 2$ for $x = 2$

Solution

$$\begin{aligned} f(x) &= x^2 - 5x + 2 \\ f(2) &= 2^2 - 5(2) + 2 \\ &= 4 - 10 + 2 \\ &= -4 \end{aligned}$$

Example 3 $f(r) = 4\pi r^2$ Evaluate $f(r)$ for $r = 5$

Solution

$$\begin{aligned} f(r) &= 4\pi r^2 \\ f(5) &= 4\pi 5^2 \\ f(5) &= 100\pi \end{aligned}$$

[A] **EXERCISES**

1 Find the value of the function $5x + 7$ when $x = 4$, $x = 2$, $x = -1$, $x = -3$

2 Find the value of the function $8 - 2x$ when $x = 1$, $x = 0$, $x = 4$, $x = -5$

3 $f(x) = 2x^2 + x - 1$ Find $f(2)$, $f(-2)$, $f(0\ 1)$

4 $f(x) = x^2 - 2x - 24$ Find $f(6)$, $f(-6)$, $f(0)$

5 $f(y) = 10 - y^2$ Find $f(3)$, $f(-3)$, $f(4)$

6 $f(r) = \frac{22}{7} r^2$ Find $f(7)$, $f(21)$, $f(14)$

7 $f(y) = y^3$ Find $f(3)$, $f(-3)$, $f(0)$

8 $f(x) = a^2 - x^2$ Find $f(a)$, $f(-a)$, $f(1-a)$

9 $f(p) = \dfrac{100}{p}$ Find $f(20)$, $f(100)$

10 $f(x) = 2x^3 - x + 6$ How much greater than $f(-1)$ is $f(1)$?

11 $f(x) = x^2 - 6x + 3$ Find the change in $f(x)$ when x changes from 2 to 3 For the interval $x = 2$ to $x = 3$, is $f(x)$ an increasing or a decreasing function? Find the change in $f(x)$ when x changes from 3 to 4 For the interval $x = 3$ to $x = 4$, is $f(x)$ an increasing or a decreasing function?

Functional Changes [A]

See if you know the following facts which are needed for the exercises below

Some Important Facts

1 If the numerator of a fraction
 is multiplied (or divided) by a number (excluding zero),
 the fraction is multiplied (or divided) by that number

2 If the denominator of a fraction
 is multiplied (or divided) by a number (excluding zero)
 the fraction is divided (or multiplied) by that number

3 If one factor of an indicated product
 is multiplied (or divided) by a number,
 the product is multiplied (or divided) by that number

Example 1 $V = lwh$ How is V affected when l is doubled?

Solution From statement 3 above we know that lwh is doubled Then V is doubled

Example 2 $V = \frac{1}{6}\pi d^3$ How does V change when d is multiplied by 5?

Solution d is used three times as a factor So $\frac{1}{6}\pi d^3$ is multiplied by 5 three times Then V is multiplied by 5^3, or 125

EXERCISES

[A]

1 How does the expression $6 \times 8 \times 4$ change when the factor 6 is multiplied by 4? when it is divided by 3?

2 How does the value of the fraction $\frac{7 \times 8}{4}$ change when the 4 is doubled? when it is halved?

3 $A = \frac{1}{2}bh$ How does A change when h is halved? trebled? cubed?

4 $S = \frac{1}{2}gt^2$ is a formula for finding the distance an object will fall from a balloon For a given place on the earth g is a constant How is S changed when t is doubled? trebled?

5 $V = e^3$ shows how to find the volume of a cube whose edge is e The edge of one cube is 4 times that of another Compare the volumes of the two cubes

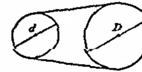
6 $C = 2\pi r$ If the radius is multiplied by $\frac{5}{6}$ what is the effect on C?

7 $V = \pi r^2h$ tells how to find the volume of a cylinder A wholesale merchant changed the size of metal cans by making them $\frac{2}{3}$ as high and their diameters $\frac{3}{4}$ as large How does the volume of the new can compare with the volume of the old one?

8 $A = \frac{1}{2}h(b + b')$ is a formula for the area of a trapezoid Is A changed when the increase in b equals the decrease in b'? Is A changed when the increase in h equals the decrease in the sum of b and b'?

9 $A = p + 03p$ shows the amount of p dollars for 1 year when the rate is 3% Factor the right member of the equation and tell how A changes when the principal is decreased 20% (made $\frac{4}{5}$ as large)

10 If two pulleys with diameters d and D inches, respectively, are connected by a belt, the pulley with diameter d makes n revolutions per minute, and the pulley with diameter D makes N revolutions per minute, then $DN = dn$ Solve the equation for N What are the independent variables in the derived equation? If n is doubled, how is N changed? How can N be increased without changing n and d? If d, n, and D are trebled, how is N affected?



11 $I = \frac{E}{R}$ is a formula used in electricity

a If E is made $\frac{2}{3}$ as large and R is not changed, how is I changed?

b If R is halved and E is constant, how is I changed?

c If both E and R are trebled, how is I affected?

12 $F = \frac{Wv^2}{gr}$ is a formula in physics

a If both W and v are doubled and g and r are constant, how is F changed?

b If r is increased by $\frac{1}{2}$, that is, made $\frac{3}{2}$ as large, how is F changed?

Making Tables from Equations [A]

A table of values that shows the change in the population of a city or state does not contain *related* values, because the change in the population does not depend upon the time element, but any table that is made from an equation contains related values. The relation between the values is expressed by the equation. Related numbers follow some law which can be expressed by an equation.

To make a table based upon an equation, we assign values to one (or more) of the variables and solve the equation for the corresponding value of the remaining variable.

EXERCISES

[A]

1 $y = 4x - 6$

a Copy and complete the table

If $x =$	-2	-1	0	1	2	3	4
$y =$	?	?	?	?	?	?	?

b Name the independent variable, the dependent variable, the constants

c How does y change when x is increased?

d Is y doubled when x is doubled?

2 $y = 10 - 5x$

a Copy and complete the table at the top of the next page

If x =	−2	−1	0	1	2	3	4
y =	?	?	?	?	?	?	?

b How does y change when x is increased?

c. Is y doubled when x is doubled?

3. $s = 16\ t^2$ expresses the law of falling bodies

a Copy and complete the table

If t =	0	1	2	3	4	5	6	7
s =	?	?	?	?	?	?	?	?

b If t is doubled, how is s changed?

c If t is increased by 1, can you tell how much s is increased?

d. If t is trebled, how is s changed?

4. $xy = 36$

a Copy and complete the table

When x =	1	2	3	4	6	?	?	?
y =	?	?	?	?	?	4	3	2

b How does y change when x is increased?

c If x is doubled, how does y change?

d. Solve the equation for x, for y

(B)

5. $P = 2\ b + 2\ h$ is the formula for finding the perimeter of a rectangle

a Copy and complete the table

b	5	10	?	?	20	15
h	4	8	8	16	4	?
P	?	?	20	42	?	54

b. What are the independent variables?

c. If both b and h are doubled, how is P affected?

d If one independent variable is halved and the other is doubled, is the value of P changed?

6 $i = prt$

a Copy and complete the table

p	200	400	400	?	200
r	04	04	?	08	?
t	3	?	3	6	6
i	?	48	96	192	48

b If r and p are constant and t is doubled, how is i affected?

c If r and t are doubled and p is constant, how is i changed?

d Complete If one factor of a product is multiplied by a number the product is ? by that number

Rational Expressions (A)

A rational expression is one in which the exponents of the letters involved are positive integers Thus a rational expression does not contain any indicated root of the unknown For example, $2x^3$, $\frac{2}{3}x^2$, $\frac{x^2}{x^2+1}$, and $x^2 - 4x + 5$ are rational expressions

Integral Expressions (A)

An integral expression is one in which the letter, or letters, involved does not appear in a denominator Thus $2x$, $\frac{x}{3}$, and $\frac{x-1}{4}$ are integral expressions, while $\frac{x^2}{x^2+1}$ is not integral Do you see that a polynomial is both rational and integral?

Degree of a Term and Function (A)

If a term contains only one variable, its *degree* equals the exponent of the variable Thus $2x$ is of the first degree and $7y^3$ is of the third degree If a term has more than one variable, its degree equals the sum of the exponents of all the variables Thus, xy is of the second degree in x and y, and x^3y^2 is of the fifth degree in x and y The definition of the degree of a term applies only to integral rational expressions

The degree of a function is the same as the degree of its highest-degree term Thus $x^2 - x + 6$ is a second-degree function of x, $3y + 1$

is a first-degree function of y, and $x^2y - 3$ is a third-degree function of x and y

A first-degree function is called a linear function, a second-degree function is called a quadratic function, and a third-degree function is called a cubic function

The degree of an equation is the same as the degree of its highest-degree term

EXERCISES [A]

State the degree of each of the following, considering both x and y as variables

1 $2x+5$	4 y^2-3y+2	7 $y=x+2$
2 $4x-3$	5 xy	8 $y=x^2-3x$
3 $x+y$	6 $10-4x$	9 $y=\frac{1}{2}x^3$

Forming Equations from Tables [A]

We have made tables from equations and now we shall make equations from tables It is not always easy to form the equation which will be satisfied by all of the sets of values in the table The task is not too difficult, however, if we know the degree of the equation

Study the following example

Example. Write the linear equation which shows the relation between x and y as shown in the table

x	0	1	2	4
y	-8	-6	-4	0

Solution. From the table we see that $y = -8$ when $x = 0$ This means that $y = ?(0) - 8$ *It means that y equals a term containing* x, decreased by 8

From the table we observe that as x increases by 1, y increases by 2 In other words, y increases twice as fast as x

From these two observations we believe the equation to be $y = 2x - 8$ Then we check the equation $y = 2x - 8$ by substituting each set of values given in the table. We find that each set of values is a solution of the equation

Then the required equation is $y = 2x - 8$

EXERCISES

[A]

Write the linear equations which express the relations between x and y as shown in the following tables, and then find the values needed to complete the tables

1

x	0	2	4	6	8	12	14
y	12	10	8	?	?	?	?

2

x	−2	2	4	6	8	−5	−7
y	−3	3	?	?	?	?	?

3

x	0	3	6	9	?	−4	−5
y	14	8	2	?	−4	?	?

[B]

Write the quadratic equations which express the relations between x and y as shown in the following tables (Either x or y or both x and y, in each equation is squared)

4

x	0	2	5	3	−4	−5	?
y	0	4	25	?	?	?	36

5

x	0	1	2	3	4	−1	−2
y	1	4	13	?	?	?	?

Graphs [A]

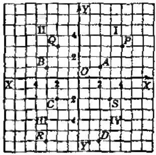
Graphs are used in mathematics chiefly to show the relation between two varying quantities, to solve problems, and to interpret data They help us to understand relations between variables

The graph of an algebraic equation is a geometric figure, such as a straight line, a curved line, a circle, or a parabola You learned in elementary algebra that the graph of a first-degree equation is a straight line For this reason first-degree equations are often called linear equations

You are already familiar with the rectangular co-ordinate system The next paragraph will enable you to recall the terms used in connection with this system

Rectangular Co ordinates [A]

On a sheet of squared or cross section paper two perpendicular lines one horizontal (XX') the other vertical (YY') are drawn XX' is called the horizontal axis and YY' the vertical axis Together they are referred to as the co-ordinate axes Their intersection is the origin (O) They divide the plane of the co ordinate system into four quadrants the first second third and fourth as shown by Roman numerals in the figure



The distance of any point such as P from the y-axis is called its x-distance or abscissa and its distance from the x axis is called its y-distance or ordinate These distances are written in parentheses in the form $(x\ y)$ and are referred to as the co-ordinates of P

x is positive in the first and fourth quadrants and negative in the second and third y is positive in the first and second quadrants and negative in the third and fourth

In the figure the co-ordinates of P are $(4\ 3)$ of $Q\ (-2\ 3)$ of $R\ (-3\ -6)$ and of $S\ (3\ -2)$

[A] **EXERCISES**

1 Write the co-ordinates of A B C and D from the figure above in the form $(x\ y)$

2 On squared paper draw a set of co-ordinate axes label them and plot the following points $A(-3\ 5)$ $B(-4\ -2)$ $C(4\ 8)$ $D(3\ -5)$ and $E(-3\ -6)$

3 On squared paper draw a set of co-ordinate axes label them and plot the following points $(5\ 0)$ $(1\frac{3}{4}\ 1)$ $(1\frac{3}{4}\ 4\frac{1}{2})$ $(-\frac{1}{2}\ 2)$ $(-4\ 3)$ $(-2\ 0)$ $(-4\ \ 3)$ $(-\frac{1}{2}\ -2)$ $(1\frac{1}{2}\ -4\frac{1}{2})$ $(1\frac{1}{2}\ -1)$ Now join them in order by straight lines

Graph of a First Degree Equation [A]

We shall assume without proof that the graph of a linear equation is a straight line

In a first-degree equation like $2x + y = 6$ for any value we assign to x there is always a corresponding value of y and conversely Any

set of values of x and y which satisfies the equation consists of the co ordinates of a point of the line We can obtain any number of points on the line Since two points determine a line, it is only necessary to find the co ordinates of two points However we usually find a third point as a check

In graphing an equation in two variables it is usually easier to find co-ordinates if we first solve the equation for one variable

Example 1 Graph the equation $2x - y = 4$

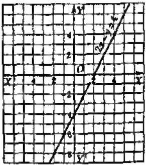
Solution Solving the equation for y we have

$$2x - y = 4$$
$$-y = -2x + 4$$
$$y = 2x - 4$$

If we let $x = -2$ then $y = -8$ If $x = 0$ $y = -4$ if $x = 2$ $y = 0$ We place these co-ordinates in a table as shown below

$y = 2x - 4$

x	-2	0	$+2$
y	-8	-4	0



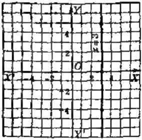
Plot the points and draw a straight line through them

Example 2 Graph the equation $x = 3$

Solution The equation $x = 3$ is equivalent to the equation $x + 0y = 3$ Since $0 \cdot y = 0$ any value assigned to y in the equation gives $x = 3$ Hence a graph is a line parallel to the y-axis and three units to the right of it

$x = 3$

x	3	3	3	etc.
y	0	5	-4	



Following the method of Example 2 you will find that the graph of $y = -2$ is a line parallel to the x axis and two units below it

An important relation between an equation and its graph is

Every point whose co-ordinates satisfy the equation lies on the graph of the equation.

And, conversely, the co ordinates of any point on the graph of an equation *satisfy the equation*

[A] **EXERCISES**

Draw graphs of the following equations

1 $x+y=5$	6 $x=6$	11 $y=2x$
2 $x-y=3$	7 $y=7$	12 $x-3y=0$
3 $2x+y=5$	8 $x-3=0$	13 $3x-4y=4$
4 $2x-y=5$	9 $y+4=0$	14 $2x+3y=12$
5 $x+3y=10$	10 $y=x$	15 $5x-2y=4$

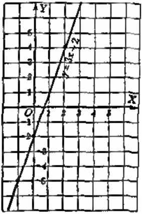
How to Graph a Function [A]

The method used to graph a function can be learned by studying the following example

Example Graph the function $3x-2$

Solution We first let y, or $f(x)$, equal the function Then we graph the equation $y=3x-2$

x	0	2	−1
y	−2	4	−5



[A] **EXERCISES**

Graph the functions

1 $2x+1$	3 $-3x+4$	5 $x-4$
2 $x+3$	4 $2x$	6 $8-x$

7. On the same set of axes draw the graphs of $2x$, $2x+6$, and $2x-4$ What do you know about the three graphs?

8. On the same set of axes draw the graphs of $\frac{1}{2}x+4$, $3x+4$, and $x+4$ Where do these graphs intersect each other?

9 a Graph the equation $f(x)=2x+3$

b Find $f(4)-f(1)$ c. Find the value of $\frac{f(4)-f(1)}{4-1}$.

Graphs of Formulas [A]

In most cases the graphs of formulas (equations) which are not of the first degree are curved lines

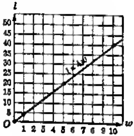
Consider the two graphs below These graphs were made from the following tables

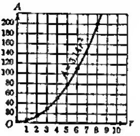
$l = 4w$

w	0	1	5
l	0	4	20

$A = 3.14r^2$

r	0	1	2	3	4	5	6	8
A	0	3.14	12.6	28.3	50	78.5	113	201





The graph of $l = 4w$ shows (within limits) the dimensions of any rectangle that is 4 times as long as it is wide The graph of $A = 3.14r^2$ gives (within limits) the area of any circle whose radius is known and the radius of any circle whose area is known

The graph of a first-degree equation, such as $l = 4w$, is a straight line For this reason we need plot only two points of the graph of a first degree equation to determine the graph We often plot a third point of the graph as a check of the other two

The graph of an equation which is not of the first degree, such as $A = 3.14r^2$, is usually a curved line Many points are needed to determine the graph when it is a curved line

To Make the Graph of an Equation Not of the First Degree

1 Make a table of sets of values satisfying the equation
2 Select suitable scales for the axes
3 Plot the points of the graph,
   using the pairs of values given in the table
4 Draw a smooth curve through the points taken in order

EXERCISES

[A]

1 a Graph the formula $i = .03p$ Let p have the values 0, 10, 20, 40, 60, 80, and 100

b How many points are needed to determine the graph?

2 a Draw the graph of $C = 3.14d$ so that it may be used to find the circumferences of circles with diameters from 0 to 6 inclusive

b Use the graph to find the circumference of a circle whose diameter is 4

c Use the graph to find the diameter of a circle whose circumference is 15

3 a Draw the graph of $I = \frac{110}{R}$, which shows the number of amperes, I, that will pass through an electric circuit when the number of volts is 110 and the resistance in ohms is R

Note that $IR = 110$ is a second-degree equation in I and R

b From the graph find I when $R = 5$

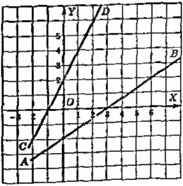
c From the graph find R when $I = 11$

4 Draw the graph of $y = 6x^2$

The x and y Intercepts [A]

When studying equations and their graphs it is often desirable to know where the graphs intersect the x and y axes

The x intercept of a line graph is the distance from the origin to the point where the graph intersects the x axis, and the y-intercept is the distance from the origin to the point where the graph intersects the y axis



In the figure shown here the x intercept of the line AB is $+3$ and the y intercept is -2, and the x intercept of the line CD is -1 and the y intercept is $+2$ What is the value of the x where the graphs intersect the y axis? What is the value of y where the graphs intersect the x axis?

> To find the x-intercept of a line,
> substitute $y = 0$ in the equation and solve for x
> To find the y-intercept of a line,
> substitute $x = 0$ in the equation and solve for y

Example. Find the y-intercept of the graph of $y = 3x - 6$

Solution $y = 3x - 6$

Substitute 0 for x, $y = 3(0) - 6$

$y = -6$, the y intercept

EXERCISES [A]

Find the x and y intercepts of the graphs of

1. $y = 3x + 6$
2. $y = -x + 4$
3. $2x - y = 7$
4. $y = 2x - 1$
5. $2y - 5x = 10$
6. $x - y = 6$
7. Which of the following equations have the same y-intercepts?

a. $y = 6x - 3$ b. $y = 4x + 3$ c. $y = -x + 3$ d. $y = 3$

The Slope of a Line [A]

No doubt you have heard some one speak of the slope of a roof, the slope of a hill, or the slope of some other object If you will associate the word "slope" with the word "steepness," you will find the following discussion easier to follow

In Fig 1, AB is the graph of $y = x$ Let a point move along AB from the origin to the point C When the point moves from O to C, its x value increases from 0 to 3 and its y value increases from 0 to 3 The x change in value is $+3$ and the y change is $+3$ The ratio $\frac{y\text{ change}}{x\text{-change}}$ is the slope of the line We define the slope of a line as follows

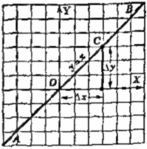


Fig 1

The slope of a line is the ratio of the change in y to the change in x We abbreviate the change in y by Δy (delta y) and the change in x by Δx Then the slope of the line $AB = \frac{\Delta y}{\Delta x} = \frac{3}{3} = 1$

In Fig 2, DE is the graph of $y = \frac{1}{3}x - 1$ As x increases from

0 *to* 3, y increases from -1 to 0 Then when $\Delta x = +3$, $\Delta y = +1$ The slope of $DE = \frac{\Delta y}{\Delta x} = \frac{1}{3}$ Note that in Fig 1 and Fig 2, y increases when x increases and that the slopes are positive

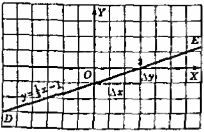


Fig 2

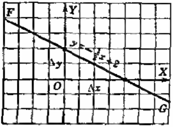
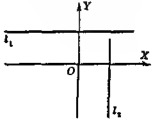


Fig 3

In Fig 3, as x *increases* from 0 to $+4$, y *decreases* from $+2$ to 0 The x change is $+4$ and the y change is -2 Then the slope of the line $FG = \frac{\Delta y}{\Delta x} = \frac{-2}{+4} = -\frac{1}{2}$ The function $-\frac{1}{2}x + 2$ is a decreasing function and its graph has a negative slope

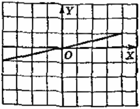


Any line which is not parallel to either of the x and y axes has either a positive or a negative slope Any line, such as l_1 in the figure, which is parallel to the x-axis, has a zero slope, and any line, such as l_2, which is parallel to the y-axis, does not have a slope This is so because $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{0}$ and division by zero is not possible

[A] **EXERCISES**

Give the slopes of the lines in exercises 1-6, assuming that each division on the axes is 1 unit

1



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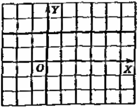

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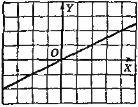

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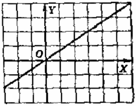

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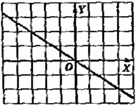

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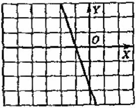


Graph the following equations and find the slopes of their graphs

7 $y = 3x - 7$	9 $2x - 3y = 6$	11 $6y = 12$
8 $y = -\frac{1}{2}x + 1$	10 $2x + y = 0$	12 $x = y$

Equations of Straight Lines[A]

The graph of a first-degree equation is a straight line, and conversely the equation of a straight line is of the first degree We can write the equation of a straight line if the conditions which determine it are known

In this text we shall consider three forms of straight line equations —the slope intercept form the point slope form and the two-point form

The Slope-Intercept Form, $y = mx + b$[A]

Let the line l have the slope m and let its graph have the y-intercept b Let the point $P(x\ y)$ denote any other point on the line except $(0, b)$

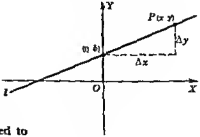
Then $m = \dfrac{\Delta y}{\Delta x}$

and $\dfrac{\Delta y}{\Delta x} = \dfrac{y - b}{x - 0}$

$$m = \frac{y - b}{x - 0}$$

$$mx = y - b$$

$$y = mx + b$$



This formula can be used to form the equation of a straight line when the slope and y-intercept are known

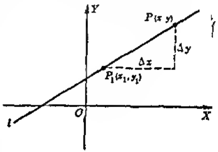
Example. Find the equation of the line whose slope is $-\frac{2}{3}$ and y-intercept is 4

Solution. $y = mx + b$, $m = -\frac{2}{3}$ and $y = 4$,
Substituting, $y = -\frac{2}{3}x + 4$, or $2x + 3y = 12$

The Point-Slope Form, $y - y_1 = m(x - x_1)$ [4]

This formula enables us to form the equation of a straight line which passes through a given point (x_1, y_1) and has the slope m

Let the point P_1 (x_1, y_1) denote the given point and let the point $P(x, y)$ denote any other point on l Let m = the slope of l



Since $m = \dfrac{\Delta y}{\Delta x}$, $m = \dfrac{y - y_1}{x - x_1}$

or $\quad y - y_1 = m(x - x_1)$

Example Form the equation of the line passing through the point $(2, -3)$ and having the slope 2

Solution $\quad y - y_1 = m(x - x_1)$

$y_1 = -3, \quad x_1 = 2, \quad$ and $\quad m = 2$

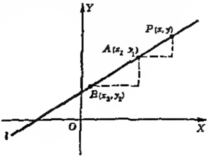
Substituting in the formula, we have

$y - (-3) = 2(x - 2)$, or $y - 2x = -7$

The Two-Point Form, $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$ [4]

This formula enables us to write the equation of a straight line when the co-ordinates of two of its points are known

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ denote two given points of the line l and let $P(x, y)$ denote a variable point of l The slope of the line segment $AP = \dfrac{y - y_1}{x - x_1}$ and the slope of the line segment $BA = \dfrac{y_1 - y_2}{x_1 - x_2}$ Since both line segments have the same slope,



$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Example Form the equation of the line passing through the points (3 4) and $(-2, 1)$

Solution $\qquad \dfrac{y-y_1}{x-x_1} = \dfrac{y_1-y_2}{x_1-x_2}$

Let $(x_1\ y_1)$ be (3, 4) and (x_2, y_2) be $(-2, 1)$

Then substituting we have $\dfrac{y-4}{x-3} = \dfrac{4-1}{3+2}$

Simplifying $\qquad 3x - 5y = -11$

EXERCISES [A]

Form the equations of the lines having the following slopes and y-intercepts

1 $m = 3, b = 0$ 3 $m = -\frac{1}{4}, b = 7$ 5 $m = -\frac{1}{2}, b = 6$

2 $m = \frac{1}{3}, b = -2$ 4 $m = \frac{2}{3}, b = 0$ 6 $m = 0, b = -3$

Form the equations of the lines passing through the following points and having the given slopes

7 $(3, 2), m = \frac{3}{2}$ 9 $(2, -1), m = -\frac{1}{2}$ 11 $(-2, -5), m = \frac{1}{4}$

8 $(2, 0), m = 1$ 10 $(2, 5)\ m = 0$ 12 $(0, 2), m = 5$

Find the equations of the lines passing through the points

13 (2, 3) and (5, 7) 16 (5, 0) and (0, 7)

14 $(-1, 2)$ and (3, 4) 17 $(-4, -4)$ and (0, 0)

15 (0, 0) and (5, 6) 18 $(-1, -1)$ and $(-4, -1)$

[B]

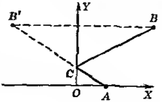
19 Write the equation of the line passing through the points (3, 1) and (5, 7) Then change the equation to the form $y = mx + b$ to give the y-intercept

20 Write the equation passing through the points $(0, b)$ and $(a, 0)$ Transform the equation so that the right member is ab Then divide both members of the equation by ab, the result is the intercept form of a linear equation

21 Write the equation of the straight line that passes through the point $(2, -3)$ and is parallel to the line $y = -\frac{1}{2}x + 6$

Note If two lines are parallel, what do you know about their slopes?

22. The co-ordinates of A, B, and C are $(2, 0)$, $(5, 4)$, and $(0, k)$ respectively. *Find the value of k that makes $AC + BC$ the smallest possible distance.*



Graphing a Linear Equation by the Slope-Intercept Method [A]

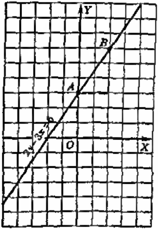
A very convenient method of drawing the graph of a linear equation consists in plotting the point where the graph intersects the y-axis and then *fixing its position by using its slope.*

Example 1. Draw the graph of $2y - 3x = 6$.

Solution. We solve the equation for y, getting $y = \frac{3}{2}x + 3$. *It is now written in the slope intercept form* $y = mx + b$. Then the slope of the graph is $\frac{3}{2}$ and the y-intercept is $+3$.

The line intersects the y-axis in the point $A(0, 3)$. Since the slope is $\frac{3}{2}$, we can find a second point of the line by starting at A and counting two units to the right and three units up. Then we draw the straight line through A and B.

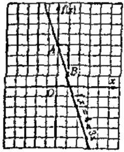
We can check the graph by determining if the co-ordinates of a third point satisfy the equation.



Example 2. Graph $f(x) = 4 - 3x$.

Solution. $f(x) = -3x + 4$, $m = -3$ and $b = 4$.

The slope of the line is $-\frac{3}{1}$ and the y-intercept *is* $+4$. *The graph intersects the* y-axis at A. Starting at A we count 1 space to the right and 3 spaces *down* to locate the point B. Then we draw the graph through A and B.



EXERCISES

[A]

Using the slope intercept method graph the following

1 $y = 2x + 1$
4 $y = 8 - 2x$
7 $2y = 5x + 2$
2 $y = 3x - 6$
5 $y = 6 + x$
8 $3y - x = 2$
3 $y - 4x = 0$
6 $y = -x$
9 $4y = 6 - 2x$

10 Graph the following equations using the same set of axes

a $y = \frac{1}{2}x + 3$
c $y = \frac{1}{2}x + 5$
b $y = \frac{1}{2}x - 6$
d $2y - x = 8$

e Copy and complete The graphs of equations having the same slope are ?

11 Graph the following equations using the same set of axes

a $y = 2x + 1$
c $y = -4x + 1$
b $y = \frac{1}{2}x + 1$
d $2y - 6x - 2 = 0$

e *Copy and complete The graphs of the equations* $y = mx + b$ where m is a variable and b is a constant ? in the same ?

12 Graph the functions $3x + 2$ and $-x + 6$ on the same set of axes Find the value of x that makes the functions equal

13 Consider the function ax How does its graph change when a increases from 0 to 5?

14 Consider the function $\frac{x}{a}$ If a is constant and x changes how does the function change when x increases?

[B]

15 On the same set of axes draw the graphs of $y = \frac{3}{2}x + 6$ and $y = -\frac{2}{3}x + 6$

16 On the same set of axes draw the graphs of $y = \frac{3}{4}x - 2$ and $y = -\frac{4}{3}x - 4$

17 If you have done exercises 15 and 16 see if you can complete the following statement The graphs of two linear equations are ? to each other if the product of their slopes equals ?

18 If you have completed the statement in exercise 17 form the equation whose graph intersects the y axis in the

point (0, 3) and is perpendicular to the line whose equation is $2x - 3y = 0$

19. If you have done exercise 10, form the linear equation whose graph intersects the y-axis in the point $(0, -1)$ and is parallel to the line whose equation is $y = -2x + 4$

Checking Your Understanding of Chapter 5

At this time you should determine if you have mastered this chapter

Be sure that you know

		PAGE
1	The meaning of functional relations	127
2	The four ways of expressing functional relations	127
3	How to make formulas from statements	128
4	How to make formulas from data	129
5	The meaning of *function*	132
6	How to evaluate a function	133
7	How to form equations from tables	136
8	How to plot points	141
9	How to graph first-degree equations	141
10	How to graph a function	143
11	How to find the x- and y-intercepts	145
12	How to find the slope of a line	146
13	How to form equations of straight lines	
	(a) by the slope-intercept method,	148
	(b) by the point-slope method,	149
	(c) by the two-point method	149
14	The meaning and proper use of the following words	

Review when you are uncertain

MATHEMATICAL VOCABULARY

	PAGE		PAGE
abscissa	141	integral expression	138
decreasing function	133	intercept	145
degree of a term	138	linear function	139
empirical	131	ordinate	141
function	132	origin	141
graph	140	rational expression	138
increasing function	133	rectangular co-ordinates	141

CHAPTER REVIEW

1 Name three ways of expressing functional relations

2 Define constant variable independent variable dependent variable

3 Name the independent variable in the formula $C = \frac{5}{9}(F - 32)$ in the formula $s = .04\,p$

4 When is one variable a function of another?

5 Write a formula for the total surface of a cube having an edge e

6 The formula $S = C + .25\,S$ gives the selling price of an article when its cost is known

a Express S as a function of C
b Express C as a function of S

7 Graph the equation $y = 5\,x - 6$

8 What is the slope of the graph of $y = 5\,x - 6$?

9 What is the y-intercept of the graph of $y = 5\,x - 6$?

10 In the equation $y = 5\,x - 6$ how much does y increase when the x increase is 4?

11 $f(x) = 3\,x^2 - 4\,x - 10$ Find $f(2)$ $f(-2)$

12 What do you know about the graphs of $y = 3\,x - 4$ and $y = 3\,x + 1$?

13 What fact do you know about the graphs of $y = \frac{1}{3}\,x + 1$ and $y = 1 - 3\,x$?

14 $V = \pi r^2 h$

a If π and r are constant and h is trebled how does V change?

b If π and h are constant and r is doubled how does V change?

15 How does the function $3\,x - 1$ change as x increases in value?

16 How does the function $8 - 3\,x$ change as x increases in value?

17 What are the x and y intercepts of the graph of $2\,y - 5\,x = -10$?

18. Form the linear equation whose graph intersects the y-axis in the point $y = -1$ and has the slope $-\frac{2}{3}$

19. Form the linear equation of the line which passes through the point $(-3, 4)$ and has the slope $\frac{2}{3}$

20 Form the equation of the line that passes through the points $(3, -4)$ and $(-2, 6)$

21. What is the slope of the graph of $y = \frac{3}{4}x - 1$?

22 Graph the equation $2x - y = 8$

23 Graph the equation $y = \frac{2}{3}x - 8$

24 $f(x) = x^2 - 6x + 3$ Find $f(-2)$

25 What do you know about the graphs of $y = 2x - 1$ and $y = 2x + 7$?

26 What do you know about the graphs of $y = 3x - 7$ and $y = 6x - 7$?

CUMULATIVE REVIEW

(A)

1. Find the simple interest on \$325 at 4% for 2 years 5 months and 15 days

2 Solve the formula $A = \frac{h}{2}(b + b')$ for h

3. Divide $x^3 - 64$ by $x + 4$

4 Combine $\frac{4x}{x-3} - \frac{2x}{x+3} - \frac{2x^2}{9-x^2}$

5 Reduce $\frac{x^2 + 2xy + y^2 - m^2}{3x + 3y - 3m}$ to lowest terms

6. If $a = 3$, $b = -5$, and $c = 2$, find the value of $(a + b)^2 - 4c^2$

7. The sum of the numerator and denominator of a fraction is 33 Find the fraction if the numerator is $\frac{3}{8}$ of the denominator.

8. Solve $\frac{2}{x+1} + \frac{31}{3x+3} = \frac{1}{6}$.

9. Divide $m(n-1) - n(n-1)$ by $n - 1$

10 Simplify $\dfrac{\dfrac{x^3 - y^3}{c^2}}{\dfrac{x^2 + xy + y^2}{4c}}$

11 Simplify $(x^2+x+1)\left(1-\frac{1}{x}+\frac{1}{x^2}\right)$

12 How much water must be added to 8 gallons of 70% solution of denatured alcohol to reduce it to a 60% solution?

13 Insert the last two terms of $4a-3b+4c$ in parentheses preceded by a plus sign

14 $f(x)=4x^3-7x$ Find $f(-2)$

15 Square $x^{2a}-y^{2b}$

16 Factor x^5+y^5

17 Graph the equation $x=\frac{10}{y}$

18 Graph the equation $y-4x=12$

19 Form the equation whose graph has the slope $\frac{2}{3}$ and intersects the y-axis in the point $y=-5$

20 Factor

a $x^2-10x+25$	d $ax+ay-bx-by$
b x^3+125	e $x^2-4x-9y^2+4$
c $8m^3-27$	f y^4-16

CHAPTER TESTS

[Test A]

1 Write a formula showing the relation between the area of a circle and the radius

2 Complete The graph of a linear equation is a ?

3 Give the degree of each of the following functions of x and y

a $2x+3$	c x^2-3x	e $2x^4-y$
b $xy-1$	d xy^2-4	f x^2-5x+3

4 Graph the function $2x-6$

5 What is the y intercept of $y=2x-10$?

6 What is the slope of the graph of $3y=6x-8$?

7 Form the equation whose graph intersects the y axis in the point $y=3$ and has the slope $+\frac{2}{3}$

8 Form the equation whose line graph passes through the points (2 1) and (−3 −6)

9 What change occurs in the function $4x - 1$ when x changes in value from 3 to -5?

10 The slope of the graph of a linear function is $\frac{3}{4}$ How much does the independent variable increase when the function increase is 9? 12

[Test B]

1. Given $f(x) = 2x^3 + 6x^2 + 10x + 3$, find $f(\frac{1}{2})$

2. Write a formula expressing the relation The area of an equilateral triangle equals $\sqrt{3}$ times one fourth the square of its side A

3 The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$ How does V change when r is doubled? trebled?

4. Graph the function $\frac{3}{2}x + 5$

5. Write the quadratic equation which expresses the relation between x and y as shown in the following table

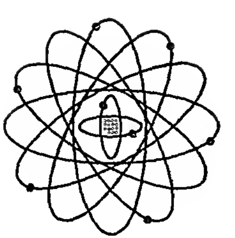
$Y = x^{2+4}$

x	0	1	2	3	4
y	3	4	7	12	19

6. Write the equation having the intercepts $(-6, 0)$ and $(0, 4)$

7. What is the slope of the line $\frac{x}{y-2} = 5$? 1/5

8. Form the equation having the slope $\frac{3}{4}$ and passing through the point $(2, 1)$



ATOMIC ENERGY

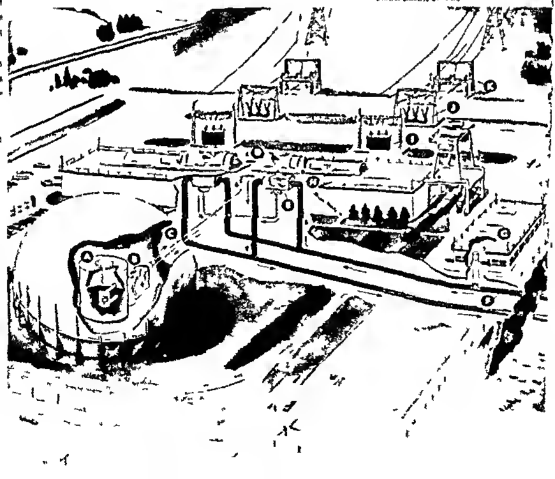
The central part of an atom is the *nucleus*. The nucleus of the hydrogen atom consists of one *proton*. The proton has a positive charge. The nucleus of any other atom consists of protons and *neutrons*. A neutron carries no charge. For each proton in the nucleus of an atom there is an electron, which has scarcely any weight and spins around the nucleus in a definite orbit. The oxygen atom has a nucleus consisting of 8 protons and 8 neutrons with 8 electrons revolving about the nucleus, as suggested in the diagram. Energy is released from the atom by fission and by fusion. In either case there is a shifting of the protons and neutrons from one atomic nucleus to another.

In 1905 Albert Einstein brought forth his famous formula $E = mc^2$, which states that when matter is destroyed it changes into energy which equals the mass of the matter multiplied by the square of the velocity of light.

The diagram shows how electricity can be made from the atom. Reactor A produces heat by controlled nuclear chain reaction. In heat exchanger B radioactive coolant transmits reactor heat to water making steam indicated by line C to drive turbines D. Condensers E collect spent steam, cool it with piped river water F from pumping station G and return it to heat exchanger. The turbines drive generators H to make electricity. Electric power from generators travels, successively through transformers I, circuit breakers J, and disconnect switches K to transmission lines that go to city.

When World War II began scientists had sufficient theoretical knowledge of the atom structure to produce the fission of the uranium atom At great expense the United States government recruited a vast army of scientists and workmen to make the atomic bomb The first experimental bomb was exploded in the desert of New Mexico on July 16, 1945 Since World War II the hydrogen bomb has been made

Although the energy of the atom was first used in war, it is now being used in other fields How will the enormous power of the atom be used in the future? Perhaps you will contribute to the present knowledge of the atom as scientists like J J Thomson, Becquerel, Lord Rutherford, Niels Bohr, Einstein, Lawrence Fermi, Lise Meitner, and others have done in the past



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CHAPTER 10

Systems
Linear Equations

In this chapter you will study
first degree equations
ring more than one unknown

You know that a first-degree equation having only one unknown has only one solution. For example, the equation $x + 6 = 8$ has just the solution $x = 2$.

A *first-degree equation having more than one unknown has an infinite number of solutions.* For example, the equation $x + y = 5$ is satisfied by any one of the following sets of values:

x	-1	1	$1\frac{1}{2}$	3	4	100
y	6	4	$3\frac{1}{2}$	2	1	-95

Try finding others. We say that such an equation is indeterminate because we cannot point to a particular set (or sets) of values which satisfy it. In this case the unknowns are variables.

Since an equation having two unknowns has many solutions, it usually happens that two such equations have a solution in common. For example, the equations $x + y = 5$ and $x - y = 1$ are both satisfied by $x = 3, y = 2$.

Graphical Solution of a Set of Two Equations [A]

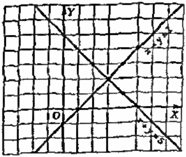
If we graph two linear equations in two variables on one set of axes, it is easy to see the common solution. It is the point where the two lines cross. For example, if we graph the two equations $x + y = 5$ and $x - y = 1$ on one set of axes, we find that they intersect in the point (3, 2), so we know that the common solution is $x = 3, y = 2$.

$x + y = 5$

x	0	2	4
y	5	3	1

$x - y = 1$

x	5	4	1
y	4	3	0



We know that the co-ordinates of the point of intersection make up the common solution because we know that every point on the graph of $x + y = 5$ has co-ordinates that satisfy the equation $x + y = 5$, and every point on the graph of $x - y = 1$ has co-ordinates that satisfy the

equation $x - y = 1$ Since the point of intersection is on both graphs, its co-ordinates satisfy both equations Conversely, we know that a common solution of the two equations is a pair of co ordinates of a point on the graphs of both lines Since these lines are distinct lines, there is only one such point—the intersection

We call equations which have a common solution consistent (or *simultaneous*)* equations

It sometimes happens that a set (or system) of consistent equations may be so related that every solution of one is a solution of the other We say that such equations are equivalent, or dependent The equations

$$x + y = 5$$
$$2x + 2y = 10$$

are equivalent because every set of values satisfying one equation also satisfies the other Do you see that $2x + 2y = 10$ is actually the same equation as $x + y = 5$? By what have the members of $x + y = 5$ been multiplied to produce $2x + 2y = 10$?

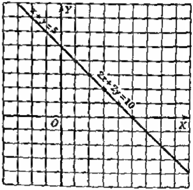
If we make a graph of equivalent equations, we get only one line, thus

$x + y = 5$

x	0	2	4
y	5	3	1

$2x + 2y = 10$

x	1	3	5
y	4	2	0



Sometimes a system of equations has no common solution The pair

$$x + y = 5$$
$$x + y = 3$$

has none because, if the sum of x and y is 5, the sum of the same x and the same y could not be 3 Equations which have no common solu-

*Some books define simultaneous equations as any two equations which are treated together for the purpose of investigating for a common solution whether there is a common solution or not.

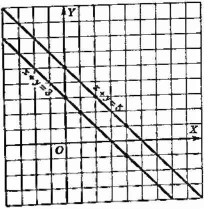
tion are called inconsistent. If we graph two inconsistent equations on the same set of axes, we get parallel lines, thus

$x+y=5$

x	0	2	4
y	5	3	1

$x+y=3$

x	0	1	4
y	3	2	−1



[A] **EXERCISES**

Examine each of the following pairs of equations and try to decide whether they are consistent or inconsistent. If they are consistent, try to decide whether they are dependent or independent. (Independent equations are distinct equations.) Check your conclusions by making a graph of each pair. If the pair has a common solution, find it.

1. $2x+y=5$
   $x-y=1$
2. $y=-x$
   $3x-2y=15$
3. $x-y=5$
   $2x+3y=10$
4. $x-y=0$
   $x+y=0$
5. $y=\frac{1}{2}x$
   $x+2y=8$
6. $y-x=1$
   $3y=2x$
7. $y=3$
   $y=5$
8. $x=-2$
   $3x+y=1$
9. $3x+4y=2$
   $x-2y=9$

Algebraic Solution of Sets of First-Degree Equations in Two Variables [A]

The algebraic solution of a set of first-degree equations is more accurate and takes less time than the graphic solution. We combine the two equations in two variables so as to form one equation in one variable. This process is called elimination. We shall study two commonly used methods of elimination and solve an example by each method.

In Chapter 16, you will learn to solve systems of equations by means of determinants.

Elimination by Addition or Subtraction[A]

When a linear equation in two variables is written in the form $ax + by = c$, it is said to be in the standard form

Example 1 Solve $\quad \dfrac{7x}{3} = 4 - \dfrac{y}{2} \quad (1)$

$$3(x+2) = 5(y+9) \quad (2)$$

Solution We first clear equation (1) of fractions,

$$\frac{7x}{3} = 4 - \frac{y}{2} \quad (1)$$

M_6 $\qquad 14x = 24 - 3y$

Writing this equation in standard form, we have

$$14x + 3y = 24 \quad (3)$$

In equation (2), removing parentheses and arranging in standard form, we have

$$3x + 6 = 5y + 45$$

$$3x - 5y = 39 \quad (4)$$

Then we solve the set of equations

$$14x + 3y = 24 \quad (3)$$

$$3x - 5y = 39 \quad (4)$$

We can make the absolute value of the coefficients of y the same in both equations if we multiply

(3) by 5 $\qquad 70x + 15y = 120$

(4) by 3 $\qquad 9x - 15y = 117$

Adding, we have $\qquad 79x = 237$

D_{79} $\qquad x = 3$

If the coefficients of y had the same sign in both equations, it would be necessary to subtract in order to eliminate y

Substituting $x = 3$ in equation (3), we obtain

$$14x + 3y = 24$$

$$42 + 3y = 24$$

$$3y = -18$$

$$y = -6$$

Therefore the solution is $x = 3$, $y = -6$

PROOF It is necessary to substitute $x = 3$, $y = -6$, in both equations in order to prove that the pair satisfies both equations

$\frac{7x}{3} = 4 - \frac{y}{2}$	$3(x+2) = 5(y+9)$
Does $\frac{21}{3} = 4 - \frac{-6}{2}$?	Does $3(3+2) = 5(-6+9)$?
Does $7 = 4 + 3$? Yes	Does $15 = 15$? Yes

The Addition-Subtraction Method of Solution

1. Reduce each equation to the standard form
2. If necessary,
   multiply each equation by such positive numbers
   as will make the coefficients of one variable
   have the same absolute value in the two equations
3. Then add or subtract (whichever is necessary in order to eliminate that variable)
4. Solve the resulting equation for the remaining variable
5. Substitute the value of this variable
   in one of the equations in standard form
   and solve for the other variable
6. Prove that the values found satisfy both equations

Elimination by Substitution [A]

If the equations contain parentheses or fractions, they are simplified before applying the method of substitution

Example Solve

$$3x - 7y = 5 \qquad (1)$$
$$2x + y = 9 \qquad (2)$$

Solution Solving equation (2) for y, we have $y = 9 - 2x$ Substituting this value of y in equation (1), we have

$$3x - 7y = 5 \qquad (1)$$
$$3x - 7(9 - 2x) = 5$$
$$3x - 63 + 14x = 5$$
$$17x = 68$$
$$x = 4$$

Substituting $x = 4$ in $y = 9 - 2x$, we obtain

$$y = 9 - 2x$$
$$y = 9 - 8$$
$$y = 1$$

Therefore $x = 4, y = 1$, is the solution of the equations $3x - 7y = 5$ and $2x + y = 9$

PROOF Substitute $x = 4, y = 1$, in the original equations

$3x - 7y = 5$ (1)	$2x + y = 9$ (2)
Does $12 - 7 = 5$? Yes	Does $8 + 1 = 9$? Yes

The Substitution Method of Solution

1. From one equation obtain the value of one variable in terms of the other.
2. Substitute this value in the other equation and solve.
3. Prove that the values found satisfy both equations.

In solving some sets of linear equations it is easier to use the addition-subtraction method, in other sets the substitution method. In the following exercises, if you are doubtful as to the easier method, solve by both methods to decide which is the easier method for any given set of equations. Be careful, some pairs are inconsistent or dependent.

EXERCISES [A]

1. $3x + 7y = 17$
   $y = 2x$

2. $5x - 2y = 19$
   $7x + 3y = 15$

3. $8x - 11y = -2$
   $y = -2$

8x + 22 = -2
8x
x = -3

4. $2x + 3y = 2$
   $15x - 7y = -29$

17x = -85
-87
x = -5
y = -4

5. $3x - 6y = 12$
   $-2x + 4y = -8$

6. $4x - 3y = 8$
   $4x - 5y = 0$

7. $2x - 4y = 6$
   $-3x + 6y = 9$

8. $2x + 78 = 5y$
   $y + 104 = 3x$

9. $x + 1.5y = 4$
   $0.9x + 2.1y = 2.1$

10. $2.3x - 4.1y = -8.6$
    $5.1x + 3.25y = 24.125$

11. $\frac{5y}{3} + \frac{7x}{6} = 68$
    $\frac{y}{4} + \frac{7x}{4} = 12$

12. $7(x - 5) + x = 3 - \frac{y}{2}$
    $\frac{x - y}{2} + y = \frac{10(x - 1)}{3} - 2$

13. $\frac{12x + 15y}{40} = 3(x - y)$
    $5y = \frac{3 - 4x}{2}$

14. $\frac{12x + 45y}{3x - 22y} = -\frac{7}{3}$
    $\frac{x + y}{8} - 1 = 0$

First-Degree Equations in More than Two Variables [A]

In order to solve a system of equations involving more than two variables, there must be as many independent equations as there are variables, and no one of the equations can be inconsistent with, or dependent on, any other equation in the system. It is not necessary

that all the variables appear in each equation. If a system of first-degree equations contains the same number of variables as equations, and can be solved, it has in general one, and only one, set of values which satisfy all equations of the system.

The following outline shows the steps in solving a system of three equations in three unknowns algebraically. The plan can be extended to include a system of any number of equations with the same number of unknowns as equations. Do you see why the graphical method is not discussed for equations with several unknowns?

To Solve a System of First-Degree Equations in Three Variables

1. Eliminate one variable from one pair of equations.
2. If necessary,
   eliminate the same variable from another pair of equations to obtain two equations involving the same two variables.
3. Solve this pair of equations in two variables.
4. Solve for the third variable
   by substituting in one of the original equations.
5. Prove by substituting the values of the three variables in the three original equations.

Example 1 Solve

$$x+y-z=4 \qquad (1)$$
$$3x-5y+4z=3 \qquad (2)$$
$$6x-7y-2z=2 \qquad (3)$$

Solution. First we shall eliminate z from (1) and (2).

M (1) by 4	$4x+4y-4z=16$	(4)
M (2) by 1	$3x-5y+4z=3$	(5)
A (4) and (5)	$7x-y=19$	(6)

Next we shall eliminate z from (1) and (3).

M (1) by 2	$2x+2y-2z=8$	(7)
M (3) by 1	$6x-7y-2z=2$	(8)
S (8) from (7)	$-4x+9y=6$	(9)

We have now reduced the original set of three equations in three variables to a set of two equations in two variables.

$$7x-y=19 \qquad (6)$$
$$-4x+9y=6 \qquad (9)$$

M (6) by 9	$63x-9y=171$	(10)
M (9) by 1	$-4x+9y=6$	(11)
A (10) and (11)	$59x=177$	
	$x=3$	

Substituting $x = 3$ in (6) we have $21 - y = 19$ or $y = 2$. Substituting $x = 3$ and $y = 2$ in (1) we have

$$x + y - z = 4 \quad (1)$$
$$3 + 2 - z = 4$$
$$z = 1$$

Therefore $x = 3$, $y = 2$, $z = 1$ is the solution of the set.

PROOF In (1) $3 + 2 - 1 = 4$
In (2) $9 - 10 + 4 = 3$
In (3) $18 - 14 - 2 = 2$

Example 2 Solve

$$x - 2y + 3z = 9 \quad (1)$$
$$3x - y = -10 \quad (2)$$
$$5y + 4z = 2 \quad (3)$$

Solution First we shall eliminate z from (1) and (3).

M (1) by 4	$4x - 8y + 12z = 36$	(4)
M (3) by 3	$15y + 12z = 6$	(5)
S (5) from (4)	$4x - 23y = 30$	(6)

We have now reduced the set to

	$3x - y = -10$	(2)
	$4x - 23y = 30$	(6)
M (2) by 23	$69x - 23y = -230$	(7)
M (6) by 1	$4x - 23y = 30$	(8)
S (8) from (7)	$65x = -260$	
	$x = -4$	

Substituting $x = -4$ in (2) we obtain $y = -2$.
Substituting $y = -2$ in (3) we obtain $z = 3$.
Therefore $x = -4$, $y = -2$, and $z = 3$ is the solution of the system.

PROOF Left to the student.

EXERCISES [A]

Solve

1. $x + y + z = 6$
   $x + y - z = 0$
   $x - y - z = 2$

2. $x + y = 4$
   $x + z = -2$
   $y + z = 8$

3. $x + 5y + 3z = 4$
   $3x - 2y + 4z = 21$
   $2x + 3y - z = -13$

4. $x + y + z = 5$
   $\frac{1}{2}y - z = 6.4$
   $-4x + 3z = 0.3$

5 $2x + 3y = 18$
$x - 4z = 7$
$y + z = 3$

6 $\frac{x}{3} + \frac{y}{2} = \frac{5}{6} - z$
$x + 1 + \frac{y}{3} = \frac{z}{2}$
$\frac{x}{3} + 1 + \frac{z}{2} = y$

7 $\frac{x}{5} + \frac{6y}{5} + 2z = \frac{7}{5}$
$x - \frac{3y}{4} =$
$\frac{x}{2} + z = \frac{1}{2}$

8 Show that the following system is inconsistent
$3x + y - 2z = 8$
$x + 2y - 3z = 6$
$2x - y + z = 1$

Systems of Fractional Equations [B]

Example Solve for x and y

$$\frac{x+4}{y} = \frac{x}{y-2}$$
$$\frac{x-4}{y} = \frac{x}{y+6}$$

Solution

$$\frac{x+4}{y} = \frac{x}{y-2} \qquad (1)$$
$$\frac{x-4}{y} = \frac{x}{y+6} \qquad (2)$$

M (1) by $y(y-2)$	$xy + 4y - 2x - 8 = xy$	(3)
M (2) by $y(y+6)$	$xy + 6x - 4y - 24 = xy$	(4)
CT in (3)	$-2x + 4y = 8$	(5)
CT in (4)	$6x - 4y = 24$	(6)
Adding (5) and (6)	$4x = 32$	
D_4	$x = 8$	
Substituting $x = 8$ in (6)	$48 - 4y = 24$	
	$y = 6$	

The solution is $x = 8$ $y = 6$

PROOF Left to the student

[B] **EXERCISES**

Reduce to linear equations in standard form and then solve

1 $\frac{2x+y}{5x-y} = \frac{1}{2}$
$\frac{x-2y+9}{4x+3y} = \frac{2}{3}$

2 $\frac{12x+y-4}{5x-4y} = 2$
$\frac{5x+2y}{3x-2y+8} = \frac{3}{2}$

3 $\dfrac{2}{x-1} + \dfrac{3}{y-1} = 0$

$\dfrac{1}{2x+5} + \dfrac{1}{y} = 0$

4 $\dfrac{x+9}{y-8} = \dfrac{x+6}{y-4}$

$\dfrac{y+4}{x-3} = \dfrac{y-4}{x+1}$

5 $\dfrac{6x}{2x+3} = \dfrac{3y-12}{y}$

$\dfrac{4x-3}{4x} = \dfrac{y}{y+4}$

6 $\dfrac{y+4}{x-4} = \dfrac{y}{x}$

$\dfrac{y-2}{x-3} = \dfrac{y+4}{x+1}$

7 $\dfrac{x+4}{x} = \dfrac{y}{y+6}$

$\dfrac{x+6}{x+2} = \dfrac{y}{y-2}$

8 $\dfrac{1}{x-2} + \dfrac{1}{y+2} = 0$

$\dfrac{3}{x+2} - \dfrac{4}{y-2} = 0$

Sets of Equations in $\frac{1}{x}$ and $\frac{1}{y}$ [A]

The equations $\quad \dfrac{6}{x} + \dfrac{10}{y} = 5$

and $\quad \dfrac{4}{x} - \dfrac{5}{y} = 1$

are not linear or first degree equations in x and y for when cleared of fractions they become $6y + 10x = 5xy$ and $4y - 5x = xy$ A set of equations of this type can be solved easily without clearing of fractions

Solution

$$\frac{6}{x} + \frac{10}{y} = 5 \qquad (1)$$

$$\frac{4}{x} - \frac{5}{y} = 1 \qquad (2)$$

Copying (1) $\quad \dfrac{6}{x} + \dfrac{10}{y} = 5$

M (2) by 2 $\quad \dfrac{8}{x} - \dfrac{10}{y} = 2$

Adding $\quad \dfrac{14}{x} = 7$

M_x $\quad 14 = 7x$

D_7 $\quad 2 = x$

Substituting $x = 2$ in (1), $\quad 3 + \dfrac{10}{y} = 5$

$3y + 10 = 5y$

$2y = -10$

$= 5$

PROOF $\frac{6}{x}+\frac{10}{y}=5$ (1) | $\frac{4}{x}-\frac{5}{y}=1$ (2)

Does $3+2=5$? Yes | Does $2-1=1$ Yes

NOTE The above set of equations can be solved when cleared of fractions by the addition subtraction method by eliminating either x or y but if you use this method be very careful because you will introduce an apparent solution which is not a true solution Remember too that division by zero is impossible

The set can also be solved by letting $A=\frac{1}{x}$ and $B=\frac{1}{y}$ solving for A and B and finally solving for x and y

[A] **EXERCISES**

Solve the following sets of equations

1 $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{6}$

2 $\frac{3}{x}+\frac{2}{y}=2$, $\frac{4}{x}-\frac{3}{y}=-\frac{1}{6}$

3 $\frac{2}{x}+\frac{1}{y}=4$, $\frac{1}{x}-\frac{2}{y}=-1$

4 $\frac{5}{a}+\frac{1}{b}=0$, $\frac{3}{a}-\frac{2}{b}=\frac{13}{10}$

5 $\frac{3}{m}+\frac{4}{n}=-\frac{1}{2}$, $\frac{7}{m}-\frac{11}{n}=9$

6 $\frac{5}{x}-\frac{2}{y}=16$, $\frac{3}{x}-\frac{4}{y}=18$

7 $\frac{3}{x}+\frac{5}{y}=\frac{3}{2}$, $\frac{2}{x}-\frac{3}{y}=-\frac{4}{15}$

8 $\frac{4}{A}-\frac{3}{B}=2$, $\frac{10}{A}-\frac{9}{B}=3$

9 $\frac{1}{x}+\frac{1}{y}=\frac{1}{5}$, $\frac{1}{x}-\frac{1}{y}=5$

10 $\frac{5}{m}+\frac{6}{n}=2$, $\frac{7}{m}+\frac{8}{n}=2$

[8]

Solve

11. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9$

$\frac{2}{x}-\frac{1}{y}+\frac{3}{z}=13$

$\frac{3}{x}+\frac{2}{y}+4=\frac{4}{z}$

12. $\frac{1}{x}+\frac{1}{y}=m$

$\frac{1}{x}+\frac{1}{z}=n$

$\frac{1}{y}+\frac{1}{z}=p$

Systems of First-Degree Literal Equations [8]

In the following equations the letters from the first of the alphabet represent known numbers and the letters x, y, and z unknown numbers. In solving these equations, remember that since the letters represent numbers, they require the same treatment that numbers would require.

Example Solve

$$ax + by = a^2 \qquad (1)$$
$$bx - ay = ab \qquad (2)$$

Solution

M equation (1) by a $\quad a^2x + aby = a^3$

M equation (2) by b $\quad b^2x - aby = ab^2$

Adding $\quad a^2x + b^2x = a^3 + ab^2$

Factoring $\quad (a^2 + b^2)x = a(a^2 + b^2)$

$D_{a^2+b^2}$ $\quad x = a$

Substituting $x = a$ in (1), $\quad a^2 + by = a^2$

$by = 0$

$y = 0$

The solution of the equations is $x = a, y = 0$

PROOF Left to the student

EXERCISES [8]

Solve the following sets of equations

1. $x + y = a - b$
   $x - y = a + b$
2. $ax - by = 0$
   $bx + ay = 1$
3. $3x + 2y = 4c$
   $x - y = 3c$
4. $x + y = 2a$
   $3x - y = 2(a + b)$
5. $2ax + by = 5a$
   $ax - 3by = -ab$
6. $x + y = -2(a + b)$
   $ay - bx = a^2 - b^2$

7. $ax + by = 2$
$bx + z = 2$
$b^2y + az = 2$

8. $x + y = ab + b^2$
$ay = bx$

9. $(m + n)x - (m - n)y = 4\ mn$
$(m - n)x + (m + n)y = 2(m^2 - n^2)$

Solving Problems by Use of More than One Unknown [A]

In Chapter 2 you studied problems which can be solved by the use of one equation in one unknown. Now you are ready to solve problems by the use of more than one unknown. In many problems you are free to decide whether to use one unknown or more than one.

In solving problems by the use of more than one unknown we follow, with a few minor changes, the steps outlined for solving problems with one unknown; that is, we

1. Determine the question or direction
2. Answer the question or follow the direction by using letters to represent the various unknowns
3. Translate the conditions of the problem into as many equations as there are unknowns
4. Solve the resulting system of equations
5. Prove that the solution obtained satisfies the conditions set forth in the problem

In addition to these general steps, specific suggestions are given from time to time to help you analyze problems and form equations.

Value Problems [A]

Example. 50 coins, consisting of quarters and dimes, are worth $9.80. Find the number of each kind of coin.

Solution Let x = the number of quarters
and y = the number of dimes

Since we have used two unknowns, we must form two equations. Study of the information shows that we can form one equation about the number of coins and one about the value of the coins. In forming the equation about values, we should express the various amounts in the same denomination. We shall

express the various amounts in this problem as cents thus x quarters $= 25x$ cents y dimes $= 10y$ cents, and \$9 80 $= 980$ cents The two equations are

	$x + y = 50$	(1)
	$25x + 10y = 980$	(2)
M (1) by 10	$10x + 10y = 500$	(3)
Copying (2)	$25x + 10y = 980$	(4)
Subtracting (4) from (3)	$-15x = -480$	
	$x = 32$, the number of quarters	
Substituting 32 for x in (1)	$32 + x = 50$	
	$x = 18$ the number of dimes	

PROOF Is the total number of coins 50? Yes
Is the total value of the coins \$9 80? Yes

[A]

1 75 coins consisting of nickels and dimes, are worth \$5 95 Find the number of each kind of coin

2 If 10 yards of percale and 12 yards of gingham cost \$10 90 when 7 yards of percale and 6 yards of gingham cost \$5 95, what is the price per yard of gingham and percale?

3 Sally's piggy bank has 27 coins, in quarters and dimes worth \$5 25 How many of each kind of coin are in the bank?

4 The coins in my pocket, consisting of nickels, dimes, and quarters are worth \$7 05 The quarters are worth \$4 60 more than the dimes and the dimes are worth 25 cents more than the nickels How many of each kind of coin are there?

5 The grocer bought oranges at 40 cents a dozen and sold them at 50 cents a dozen He lost 30 oranges from spoilage If he made \$3 75 on the oranges how many oranges did he buy?

6 Mrs Hunter bought \$5 00 worth of stamps in 1¢, 3¢, and 8¢ denominations The value of the 8¢ stamps was 4 times the value of the 1¢ stamps, and the 3¢ stamps were worth 20 cents less than twice the value of the 8¢ stamps How many stamps of each denomination did she buy?

7 The admission price to the game between the Hornets and Bears was 60¢ for adults and 15¢ for children If the gate receipts from 7240 paid admissions amounted to \$3394 50, how many adults and how many children attended the game?

8 If Sam buys 6 golf balls at one price, he will have 30 cents left over, but if he buys 5 golf balls at another price, he will have 50 cents left over. He uses all his money to buy 3 of each kind of golf balls What was the price of each kind of golf ball?

9. A druggist sold 3 bars of soap and two tubes of tooth paste to one customer for 69 cents and made 18 cents on the sale He sold 4 bars of the same soap and one tube of the same tooth paste to another customer for 50 cents and made 7 cents on the sale How much did a bar of soap and a tube of tooth paste cost the druggist?

Investment Problems [A]

Example. A man invested \$5000, part at $2\frac{1}{2}\%$ interest per year and the rest at 4% How much did he invest at each rate if his income from both investments was \$152 per year?

Solution Let $x =$ the number of dollars invested at $2\frac{1}{2}\%$

and $y =$ the number of dollars invested at 4%

$0.025x =$ income from amount invested at $2\frac{1}{2}\%$

$0.04y =$ income from amount invested at 4%

Then	$0.025x + 0.04y = 152$	(1)
and	$x + y = 5000$	(2)
M (1) by 1000	$25x + 40y = 152{,}000$	(3)
M (2) by 40	$40x + 40y = 200{,}000$	(4)
S (4) from (3)	$-15x = -48{,}000$	
D_{-15}	$x = 3200$	
From (2)	$y = 1800$	

\$3200 is invested at $2\frac{1}{2}\%$ and \$1800 at 4%

PROOF $\$3200 \times 0.025 = \$80$

$\$1800 \times 0.04 = \$72$

Total income $= \$152$

[A] **EXERCISES**

1 \$4000 is invested, part at $1\frac{1}{2}\%$ and the remainder at 3% How much is invested at each rate if the total income from both investments is \$78?

2 Silas Silver invests twice as much money at 2% as he invests at 3% If the amounts were interchanged, his annual income would be increased by \$24 How much does he invest at each rate?

3 Mr and Mrs Mills invested \$25,000, part at 3%, part at 4%, and the remainder at 5% Their income from all the investments is \$990 The income from the 5% investment is \$210 a year more than the income from the 4% investment How much money have they invested at each rate?

4 On two investments of \$2500 and \$3500 Grandpa Sims receives a yearly income of \$197 50 The rate of interest on the larger of the two investments is $\frac{1}{2}$% more than that on the other What is the rate on each sum of money invested?

5 Mr Telford is investing some money in bonds He receives 1% more interest on an investment of \$3600 than he does on a second investment of \$2800 If he receives annually \$196 from the two investments, what is the rate of interest on each investment?

6. A man has \$6000 invested, part at 5%, part at 3%, and the remainder at 2% His annual income from the 5% and 3% investments is \$135, from the 5% and 2% investments, \$125, and from the 3% and 2% investments, \$110 How much money has he invested at each rate?

Digit Problems[A]

Our number system is based on 10 In the number 4362, 4 is the thousands' digit, 3 the hundreds' digit, 6 the tens' digit, and 2 the units' digit The number means $4000 + 300 + 60 + 2$ If in a three-digit number we represent the units' digit by x the tens' digit by y, and the hundreds' digit by z, then $100z + 10y + x$ represents the number

Example. The sum of the digits of a two-digit number is 10 If the digits are interchanged, the new number is 36 less than the original number What is the original number?

Solution. Let $x =$ the units' digit
and $y =$ the tens' digit
Then $10y + x$ represents the number,
and $10x + y$ represents the new number when the digits are interchanged

Then $$x + y = 10 \qquad (1)$$
and $$(10y + x) - (10x + y) = 36 \qquad (2)$$

Simplifying (2) $10y + x - 10x - y = 36$

$-9x + 9y = 36$

or $x - y = -4$ (3)

Solving the set $x + y = 10$ (1)

$x - y = -4$ (3)

By addition $2x = 6 \quad x = 3$

By subtraction $2y = 14 \quad y = 7$

Therefore the number is 73

PROOF $7 + 3 = 10 \quad 73 - 37 = 36$

EXERCISES

[A]

1 The sum of the digits of a two digit number is 6 If 18 is subtracted from the number the digits are reversed What is the number?

2 The sum of the digits of a two digit number is 7 If 45 is added to the number the digits will be interchanged Find the number

3 In a two digit number the tens digit is 2 more than the units digit The number is 7 times the sum of its digits What is the number?

4 The difference of the digits of a two digit number is 3 If the digits are interchanged the sum of the new number and the original number is 121 Find the original number

5 The units digit exceeds the tens digit of a two digit number by 7 If the digits are interchanged the resulting number is 9 times the sum of the digits What is the number?

6 The sum of the three digits of a number is 9 If the order of the digits is reversed the number is decreased by 198 If the units and tens digits are interchanged the number is decreased by 9 Find the number

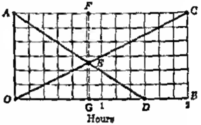
7 If the hundreds digit and the units digit of a three digit number are interchanged the number is increased by 693 If the tens digit and the units digit are interchanged the number is increased by 54 The sum of the three digits is 11 Find the number

8 The sum of the three digits of a number is 15 The hundreds digit is 3 less than the sum of the other two If the hundreds digit is interchanged with the tens digit the number is decreased by 360 What is the number?

Graphical Solution of Verbal Problems [A]

The method you have used in solving problems in this chapter is known as the analytical method Many types of problems may also be solved by the graphical method Usually the analytical method is quicker and more accurate However in some cases it is advantageous to use the graphical method

Example 1 Jim can mow his father s lawn in $1\frac{1}{2}$ hours, but it takes his younger brother Tom 2 hours to mow the lawn How long will it require both boys working together to mow the lawn?



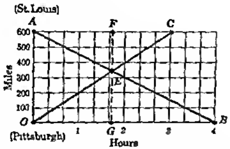
Solution On the chart at the right the whole job of mowing the lawn is represented by the vertical distance OA and two hours by the horizontal distance OB The distance OB is divided into 12 small parts each representing 10 minutes and the distance OA is divided into 6 parts each representing one-sixth of the whole job

Line AD represents the fact that Jim can mow the whole lawn in $1\frac{1}{2}$ hours and line OC represents the fact that Tom can mow the whole lawn in 2 hours Observe that a line is drawn through E the intersection of AD and OC On this line EF represents the part of the mowing Jim has completed and GE the part Tom has completed Since $EF + GE = OA$ the graph shows that when both boys work the job is completed at the time when lines AD and OC intersect Since OG represents about 52 minutes it takes the boys together about 52 minutes to mow the lawn

A larger and more accurate graph will show the time to be $51\frac{3}{7}$ minutes

Example 2 One airplane can fly from St Louis to Pittsburgh in 4 hours Another can fly from Pittsburgh to St Louis in 3 hours If they leave at the same time how soon will they meet?

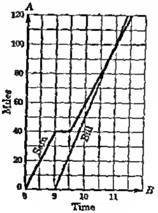
Solution On the chart below, the distance between St Louis and Pittsburgh is represented by vertical distance OA and the 4 hours by horizontal distance OB



Line AB represents the fact that one plane can fly from St Louis to Pittsburgh in 4 hours and the line OC represents the fact that the other plane can fly from Pittsburgh to St Louis in 3 hours E represents the meeting point, and OG the elapsed time to the point of meeting It is approximately 1 hour and 43 minutes EF represents the distance flown by the plane from St Louis and EG represents the distance flown by the plane from Pittsburgh

Example 3 Sam Black left Dayton at 8 A M, driving east at 40 miles per hour After an hour he stopped for 30 minutes for repairs on his truck, and then continued his journey at 40 miles per hour Bill Green left Dayton at 9 A M, driving east at 50 miles per hour along the same road When did he catch up with Sam?

Solution In the chart at the right, the hours are marked off on line OB, and the miles on line OA Observe that Bill catches Sam at about 11 A M How far were they from Dayton at that time?



EXERCISES

[A]

1 Two automobiles start toward each other at the same time from two cities 360 miles apart If one automobile travels 40 miles an hour and the other 50 miles an hour, how soon will they meet?

2 One pump can fill a tank in 5 hours and another can fill the same tank in 4 hours How long will it take both pumps to fill the tank?

3 One automobile traveling 35 miles an hour leaves Toronto at 8 A M Another automobile traveling 50 miles an hour on the same road leaves Toronto at 10 A M How soon will the second automobile overtake the first?

4 One pump can fill a tank in 6 hours and another pump can empty the same tank in $2\frac{1}{2}$ hours How long will it take to empty a full tank if both pumps are working?

5 One airplane makes a 2000 mile nonstop flight traveling 200 miles an hour Another airplane whose rate is 250 miles an hour starts one hour after the first plane on the same flight If the second airplane must stop 30 minutes to refuel at an air field 1200 miles from the starting point when will it overtake the first plane?

MISCELLANEOUS PROBLEMS

[A]

1 If Bill were 6 years older he would be one half as old as his father, and if he were 2 years younger he would be one fourth as old as his father What are the ages of Bill and his father?

2 Fred forgot to record the temperature of a liquid he had used in a chemistry experiment but he remembered noticing that the Fahrenheit temperature was numerically 5 times as great as the corresponding centigrade temperature What were the centigrade and Fahrenheit temperatures of the liquid? Remember $F = \frac{9}{5}C + 32$

3 If five times the width of a rectangle equals four times the length and the perimeter is 54, what are the width and the length?

4 A motorist traveling at a certain rate can go from one town to another in 5 hours If he travels 10 miles an hour

faster, he can drive the same distance in 1 hour less time How far is it between towns?

5 A boy weighing 75 pounds sits on a seesaw 6 feet from the fulcrum, and another boy weighing 105 sits 8 feet from the fulcrum, but on the other side To balance the board, where should a third boy sit if he weighs 60 pounds?

6 A goldsmith has two alloys that are respectively 50% and 80% pure gold How many grains of each must he use to make 300 grains of an alloy that is 72% pure gold?

7 The sum of three numbers is 62 If the second is half of the first and the third is 6 more than the second, what are the three numbers?

Checking Your Understanding of Chapter 6

Before leaving this chapter make sure that you can

1 Solve a set of two first-degree equations in two unknowns
  (*a*) Graphically,
    provided that *the solution can be read accurately* from a graph (p 161)
  (*b*) Algebraically
    by both the addition-subtraction method (p 164) and the substitution method (p 165)
  If you expect to do better than average work, you will also want to be able to solve sets involving fractions (p 169) and sets involving literal numbers (p 172)

Do you need to review?

2 Solve algebraically a set of first-degree equations with three unknowns (p 166)

3 Translate the words of a verbal problem into a set of equations *with as many equations as unknowns* (pp 173–181)

4 Spell and use the following words

MATHEMATICAL VOCABULARY

consistent (p 162)	inconsistent (p 163)
dependent (p 162)	independent (p 163)
equivalent (p 162)	simultaneous (p 162)

CHAPTER REVIEW

[A]

Solve graphically

1 $3x - y = 7$
$x + 2y = 7$

2 $x + y = 10$
$2x = 7$

Solve algebraically

3 $3x - 4y = 16$
$5x + 6y = 14$

4 $\frac{x}{3} + \frac{y}{2} = 3$
$\frac{x}{4} - \frac{y}{3} = -6\frac{1}{4}$

5 $2x + 3y = 4$
$x + 2y = 0$

6 $3x - 2y + z = 13$
$2x + 3y + 5z = 0$
$5x + y = 13$

Tell whether the following sets of equations are consistent or inconsistent. If consistent, tell whether they are dependent or independent.

7 $3x - 4y = 3$
$4x - 3y = -21$

8 $2x - 5y = 10$
$6x - 15y = 8$

[B]

Solve algebraically

9 $mx + ny = 1$
$mx - ny = 1$

10 $\frac{x+2}{y} = \frac{x}{y-1}$
$\frac{x-3}{y} = \frac{x}{y+3}$

11 $cx + y = c^2 - 2c$
$x - ay = 2ac + c$

12 $\frac{3}{x} + \frac{2}{y} = 1\frac{1}{3}$
$\frac{9}{x} - \frac{12}{y} = 1$

CHAPTER TESTS

[Test A]

Part I Numerical Equations

1 Solve graphically $3x - y = 8$
$x + 2y = 5$

2 Tell whether the following sets of equations are consistent or inconsistent. If consistent, tell whether they are dependent or independent.

a $x + 2y = 3$
$x + 2y = 12$

b $x - 2y = 3$
$x + 2y = 12$

c $x + 2y = 3$
$4x + 8y = 12$

Solve

3 $x + 2y = 5$
$2x - 5y = -8$

4 $2x - 3y = 16$
$3x + 2y = 11$

5 $5x - 3y = 21$
$x + 2y = -1$

6 $x - 3y = 0$
$5x - y = -14$

7 $\frac{5y}{4} + \frac{7x}{3} = 12$
$y - \frac{5x}{4} = \frac{1}{4}$

8 $x + y - z = -7$
$3x - 2y + 3 = 24$
$2x + 3y - 5z = -32$

Part II Verbal Problems

1 Last month the Greenville Civic League sent 4000 pieces of mail to people in the community If some pieces required 2¢ postage and the remainder 3¢ postage how many of each kind were sent if the total postage bill was $95?

2 Mr Johns saves 50 cent pieces and Mrs Johns saves quarters During May Mrs Johns saved 5 more coins than Mr Johns If together they saved $23 75 how many of each kind of coin was saved?

3 The sum of the digits of a two digit number is 13 If digits are reversed the resulting number is 45 less than the original number What was the original number?

4 The Clayton Alumni Association annually awards a $125 scholarship to an outstanding senior If the money for the award is the interest obtained from a $4000 investment which is divided between a savings certificate paying 3% interest and a bond paying $3\frac{1}{2}$% interest how much is invested at each rate?

5 A man has twice as many dimes as he has pennies and he has 3 fewer nickels than dimes If he has $2 64 in all how many coins of each kind does he have?

6 A man has $800 more invested at 4% than he has invested at 3% If his annual income from the two investments is $130 how much money does he have invested at each rate of interest?

[Test B]

Part I Numerical Equations

Solve

1 $\dfrac{x+y}{2} = 5$
$2(4y - x) = \dfrac{3x - y}{6} + 7$

2 $ax + by = 2ab$
$bx - ay = a^2 + b^2$

3 $\dfrac{3}{3x - y - 2} = \dfrac{4}{x + 2y}$
$\dfrac{9}{9x + 5y + 4} = \dfrac{1}{2x - y}$

4 $3x + 4y + 2z = 3$
$5x - 2y - 13z = 3$
$4x + 3y - 3z = 6$

5 $\dfrac{3}{x} + \dfrac{4}{y} = \dfrac{7}{24}$
$\dfrac{8}{x} - \dfrac{6}{y} = \dfrac{1}{12}$

6 $\dfrac{1}{2x} + \dfrac{2}{3y} = 15$
$\dfrac{4}{5x} + \dfrac{6}{5y} = 20$

7 Solve graphically

$$3x = y$$
$$3x - 5y = -24$$

Part II Verbal Problems

1 One alloy of copper and silver is 25% silver and another is $37\frac{1}{2}$% silver How much of each alloy should be taken to obtain 200 pounds of an alloy that is 30% silver?

2 A lever balances when weights of 120 pounds and 160 pounds are at the ends of the lever If 40 pounds are added to the heavier weight, the fulcrum must be moved 1 foot closer to it in order to balance the lever How long is the lever?

3 The sum of the digits of a three-digit number is 9 The units digit is twice the sum of the other two digits When the order of the digits is reversed, the number is increased by 495 Find the number

126

4 A man invests \$5000, part at 5% and the remainder at 4% If the annual income from both investments is \$235, what is the amount invested at each rate?

8 5

5 Albert and Bob ran a race of 440 yards In the first trial Albert gave Bob a start of 65 yards and won by 20 seconds In the second trial Albert gave Bob a start of 34 seconds and Bob won by 8 yards Find the rates of Albert and Bob in yards per second

6 Mr Greene sold 4 pounds of roast and 3 pounds of steak for \$5 56 to one customer To another customer he sold 3 pounds of roast and 4 pounds of steak and the bill was \$5 43 How much did he charge for each kind of meat?

Mathematics in chemistry

If you are planning a vocation in the field of chemistry, you should take all the chemistry courses given in your high school, since no one can learn too much about the subject. For the usual one-year course in high-school chemistry only one year of algebra is needed. The main requirement is that you be able to solve proportions and equations.

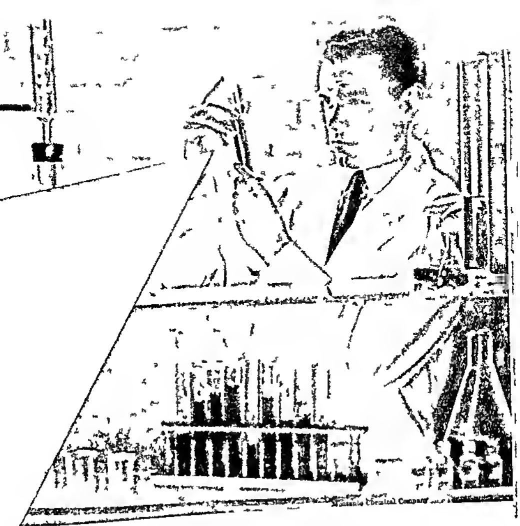
Bettmann Archive

Marie Curie

Edward Morley

Carl Scheele

Edward Acheson



Monsanto Chemical Company

If you wish to go far as a chemist you should have at least four years of college training with a major in chemistry. For advanced courses in chemistry three years of college mathematics are required by many universities. The following high school subjects are suggested as preparation for a student expecting to major in chemistry while in college:

- English (4 years)
- Chemistry (1 to 2 years)
- Algebra (1½ years)
- Plane geometry (1 year)
- Physics (1 year)

- Solid geometry (½ year)
- Foreign language (2 to 4 years)
- Social studies (2 years)
- Shop
- Mechanical drawing

CHAPTER 7

Ratio, Proportion, and Variation

In this chapter you study relationships expressed as quotients

Ratio[A]

The ratio of one quantity to another like quantity is the indicated numerical value of the first quantity divided by the numerical value of the second. Thus the ratio of 7 feet to 10 feet is $\frac{7}{10}$, and the ratio of 8 dozen to 2 dozen is $\frac{4}{1}$. The ratio of 7 inches to 2 feet is $\frac{7}{24}$ since the measures must be of the same unit. One cannot find the ratio of two unlike quantities, such as the ratio of 6 dollars to 8 bushels.

Besides the fraction bar, the colon and the division sign can be used to denote ratio. Thus $\frac{2}{3}$, $2 : 3$, and $2 \div 3$ express the ratio of 2 to 3. The ratio $\frac{2}{3}$ can be read "the ratio of 2 to 3."

[A] **WRITTEN EXERCISES**

Express the ratio of the first quantity to the second, and reduce each ratio to lowest terms.

1. 7 and 28
2. 16 and 32
3. 4 feet and 12 feet
4. 2 yards and 1 foot
5. 10 minutes and 3 hours
6. $3x$ and $15x$
7. $a^2 - 2a - 8$ and $a + 2$
8. $x^3 + 2x$ and $x^2 - 4$
9. $a^2 - x^2$ and $x - a$
10. $1 + x$ and $1 + x^3$
11. Find the ratio of one angle of an equiangular triangle to 180°.
12. Find the ratio of 20′ to 1°.
13. Find the ratio of an exterior angle of a triangle to the sum of the two nonadjacent interior angles.
14. What symbol is used to denote the ratio of the circumference of a circle to the diameter?
15. One number is $2x$ and another is $5x$. What is the ratio of the two numbers?
16. The ratio of two numbers is $3 : 4$. If $3x$ represents the smaller number, how can you represent the larger?
17. How does the ratio $\frac{3}{1+x}$ change when x is increased?
18. How does the ratio $\frac{4}{5-x}$ change when x is decreased?
19. How does the fraction $\frac{4x}{1-x}$ change when x is increased?

HINT. Change the fraction to a mixed number.

PROBLEMS

[A]

1 The sum of two numbers is 126 Find the numbers if their ratio is 2 5 (Let $2x$ = the smaller number Then the larger is $5x$)

2 Separate \$450 into two parts so that the parts will have the ratio 4 5

3 One of two boys can do $\frac{2}{3}$ as much work as the other On the basis of their abilities to do the work how should they divide \$50 75 which they receive for weeding a field of onions?

4 A certain power plow can break up as much ground as three horse-drawn plows Mr Davis uses a power plow and Mr Frank supplies two horse-drawn plows in plowing a field How should these two men divide \$80 received for plowing a field?

5 The ratio of the width and length of a rectangular field is 5 6 Find the dimensions of the field if its perimeter is 220 rods

6 The sides of a triangle have the ratio 2 3 4 Find the sides if the perimeter is 108 inches (Let $2x$ $3x$, and $4x$ represent the lengths of the sides)

7 The ratio of the sides of a triangle is 3 4 5 Find the lengths of the sides if the perimeter of the triangle is 138 inches

8 The line segment AB is 10 inches long The ratio of AC to AB is 0 618 Find AC and CB

Proportion [A]

A proportion is an equation whose two members are ratios Thus, $\frac{x}{3}=\frac{4}{7}$, $\frac{x+1}{2}=\frac{3}{x}$, and $\frac{1-c}{c+1}=\frac{2}{3}$ are proportions The equation $\frac{x+1}{3}=x+\frac{1}{4}$ is not a proportion, but it can be changed into the proportion $\frac{x+1}{3}=\frac{4x+1}{4}$

Since a proportion such as $\frac{a}{b}=\frac{c}{d}$ was formerly written in the form a b c d or a $b = c$ d, the terms a and d are called the

extremes and the terms b and c are called the means, also a is the *first term* b is the *second term* c is the *third term* and d is the *fourth term* of the proportion

Since d is the fourth term in the proportion $\frac{a}{b}=\frac{c}{d}$ it is called the fourth proportional to a b and c

When a number is used for both the second and third terms of a proportion it is called the mean proportional between the first and fourth terms Thus in the proportion $\frac{4}{x}=\frac{x}{16}$ x is the mean proportional between 4 and 16

If b is the mean proportional between a and c then c is called the third proportional to a and b Thus in the proportion $\frac{a}{b}=\frac{b}{c}$ c is the third proportional to a and b

The following theorem offers a quick method of clearing some proportions of fractions

Theorem In any proportion the product of the means is equal to the product of the extremes

Given $\frac{a}{b}=\frac{c}{d}$

To prove $ad=bc$

1 $\frac{a}{b}=\frac{c}{d}$	1 Given
2 $\frac{bd}{1}\left(\frac{a}{b}\right)=\frac{bd}{1}\left(\frac{c}{d}\right)$	2 If equals are multiplied by equals the products are equal
3 $ad=bc$	3 A quantity can be substituted for its equal

[A] **EXERCISES**

1 Form a proportion using the ratios $\frac{14}{18}$ and $\frac{21}{27}$

2 Which of the following are proportions?

a $\frac{3}{4}=\frac{4}{5}$ c $\frac{ab}{1}=\frac{abc}{c}$ e 9 8 = 8 7

b $\frac{7}{8}=\frac{21}{24}$ d $\frac{6}{9}=\frac{18}{27}$ f 5 8 15 24

3 Clear the equations of fractions by the use of the theorem above and solve

a $\frac{x}{7}=\frac{3}{8}$ b $\frac{2x-1}{3}=3$

c $\frac{x-1}{4} = 2$

d $\frac{3c+2}{2} = \frac{c-1}{1}$

e. $\frac{2x+3}{4} = \frac{x-5}{3}$

f $\frac{3y-4}{8} = \frac{4y+8}{4}$

4 Find the fourth proportional to

a 2, 3, and 9

b 10, 4, and -5

c $\frac{1}{2}$, $\frac{1}{3}$, and 36

d $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{5}$

e 4, -6, and $+6$

f c, a, and b

5 Find the mean proportional between

a 1 and 4

b 9 and 25

c $\frac{1}{4}$ and 1

d a and $9a$

e 27 and 3

f $\frac{1}{8}$ and $\frac{1}{2}$

6 Find the third proportional to 3 and 6

PROBLEMS

(A)

1 If 10 pounds of loin roast cost \$5.50, what will 7 pounds of the roast cost?

HINT The cost is proportional to the weight, $\frac{10}{7} = \frac{5.50}{x}$.

2 If $4\frac{1}{2}$ pounds of butter cost \$2.79, what will $6\frac{3}{4}$ pounds of butter cost?

3 If 140 feet of link fencing cost \$133, find the cost of 98 feet

4 A farmer sold hogs weighing 1746 pounds for \$401.58 At the same price per pound how much will he receive for hogs weighing 4124 pounds?

5 If the circumference of one circle is 250 inches, find the circumference of another circle whose diameter is $3\frac{1}{2}$ times that of the first one $\frac{C_1}{C_2} = \frac{d_1}{d_2}$

6 If 4 pints of lemonade will serve 2 boys twice, how much lemonade will be needed to serve 120 boys twice?

7 A man received \$51.75 for 45 hours' work At the same rate what will he receive for 41 hours' work?

8 If 3 bars of candy sell for 13 cents, what is the selling price of 24 bars?

Geometry Problems[A]

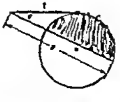
You should recall the following facts which you learned in plane geometry

1 The corresponding sides of similar ($\sim$) polygons are proportional that is in the triangles below $\frac{m}{m}=\frac{n}{n}=\frac{p}{p}$

2 The areas of two similar polygons are proportional to the squares of their corresponding sides In the diagrams below

$$\frac{T}{T}=\frac{a^2}{a^2}=\frac{b^2}{b^2}=\frac{c^2}{c^2}=\frac{d^2}{d^2}$$

3 The tangent to a circle from an external point is the mean proportional between the whole secant and its external segment In the figure



$$\frac{e}{t}=\frac{t}{s} \text{ or } t^2=es$$

EXERCISES [A]

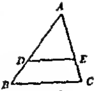
1 The dimensions of one rectangle are 60 inches and 96 inches Find the longest side of a similar rectangle if its shortest side is 10 inches

2 The lengths of the sides of a triangle are 15 inches 18 inches and 24 inches Find the length of the longest side of a similar triangle whose shortest side is 10 inches

3 If a tree 92 feet high casts a shadow 54 feet long how long a shadow will a telephone pole cast at the same time the pole being 24 feet high?

In the diagram at the right $DE \parallel BC$. By geometry

$$\triangle ADE \sim \triangle ABC$$



Exs. 4-8

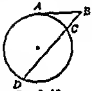
4 If $AD = 8$, $AB = 20$, and $AE = 6$, find EC.

5 Find AE when $EC = 24$, $AD = 24$, and $BD = 16$.

6 Find DE when $BC = 18$ inches, $AD = 10$ inches, and $AB = 15$ inches.

7 Find BC when $DE = 18$, $DB = 10$, and $AD = 20$.

8 Find the area of $\triangle ADE$ when the area of $\triangle ABC$ is 90 square inches, $AD = 10$ inches, and $BD = 4$ inches.



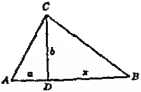
Exs. 9, 10

9 Find BD in the diagram at the right when $AB = 10$ and $BC = 4$.

10 Find DC when $AB = 8$ and $BC = 4$.

11 The area of a polygon is 176 square inches. Find the area of a similar polygon if one of its sides is $\frac{3}{4}$ as long as the corresponding side of the first polygon.

12 ABC at the right is a right triangle having $CD \perp AB$, the hypotenuse. It is proved in geometry that $\frac{a}{b} = \frac{b}{x}$.



a Find x if $a = 4$ and $b = 12$.

b Find a when $x = 3$ and $b = 9$.

c Find x if $a = 2.5$ and $b = 1.5$.

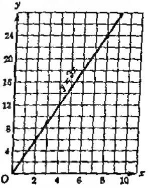
Direct Variation [A]

If two numbers change so that one number always equals a constant times the other, either number is said to vary directly as the other. If $x = cy$ and c is a constant, then x varies directly as y and y varies directly as x. Since $x = cy$ and $\frac{x}{y} = c$ are equivalent equations we can define direct variation as follows:

If two numbers change so that their ratio is constant, the numbers vary directly.

Let us graph the equation $y = 3x$, in which y and x vary directly

x	0	3	8
y	0	9	24



Direct variation is expressed by a linear equation, whose graph is a straight line Both the equation $y = 3x$ and its graph show that y increases when x increases and that y increases 3 times as fast as x Notice that the slope of the graph is 3 In general, if $y = kx$, y increases k times as fast as x

If one variable in a direct variation increases, the other variable increases

Inverse Variation

If two numbers change so that their product is constant, they vary inversely Thus if $xy = c$ when c is a constant, x and y vary inversely If $xy = c$, then $x = \frac{c}{y}$, and $y = \frac{c}{x}$ Then we can define inverse variation as follows

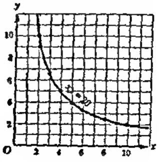
If two numbers change so that one equals a constant divided by the other, the numbers vary inversely

Let us graph $xy = 20$, which states that x and y vary inversely

x	2	3	4	5	6	8	10
y	10	67	5	4	33	25	2

Since x and y in the equation $xy = 20$ are of the first degree, the expression xy is of the second degree in x and y As you know, an equation that is not of the first degree usually has a curved-line graph

The graph of $xy = 20$ is a curve called an equilateral hyperbola It has two distinct branches Only the branch in the first quadrant is shown Either the graph or the equation shows that one variable decreases when the other increases



If one variable in an inverse variation increases, the other variable decreases.

Joint Variation [A]

The formula $F = \frac{Wv^2}{320}$ tells how to find the centrifugal force, F, of a body moving around a circle having a radius of 10 feet In this formula F varies directly as the weight, W, and the square of the velocity, v F varies jointly as W and v^2

In the formula $A = lw$, A varies jointly as l and w, and in the formula $V = \frac{1}{3}bh$ V varies jointly as b and h

One variable varies jointly as two or more variables when it varies as the product of these variables

EXERCISES [A]

1 Does the cost of a number of articles of the same kind vary directly or inversely as the number of articles?

2 Does the number of gallons of gasoline used on an automobile trip vary directly or inversely as the number of miles of the trip?

3 Given that x men can build a house in y days How do x and y vary?

4 The area of a rectangle is 24 If the base of the rectangle is b and the height of it is h, how do b and h vary?

5 The annual interest on a variable sum of money is \$24 Is the rate a constant or a variable? How do the principal and rate vary?

6 In a formula used in electricity, E varies jointly as I and R Write the formula, the constant being 1

Tell how the variables vary in the following formulas and equations

7 $\frac{1}{y} = 6x$

8. $xy = 20$

9 $4y = x$

10 $c = 2\pi r$

11 $rs = 12$

12 $F = \frac{4}{R}$ FR = 4

13 $P = 3r$

14 $i = pr$

15. $\frac{m}{n} = 4$

16. $\frac{x}{3} = y$

17. $ab = 4$

18 $y = \frac{10}{z}$

Write each of the following statements in algebraic language, using k as a constant

19. The circumference of a circle varies directly as the diameter

20. The volume of an enclosed gas at constant temperature varies inversely as the pressure

21 The distance an object will fall varies directly as the square of the time (Let s = the number of feet in the distance and t = the number of seconds the object falls)

22 The volume of a sphere varies as the cube of its diameter

23 The surface of a cube varies directly as the square of one of its edges What is the value of k in this formula?

Example 1 y varies directly as x, and $y = 80$ when $x = 16$ Find x when $y = 100$

Solution 1 y varies directly as x

Then $y = kx$ k being a constant
Then $80 = k \cdot 16$ and $k = 5$
Then $y = 5x$
When $y = 100$, $100 = 5x$ and $x = 20$

Solution 2 Since y varies directly as x

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}$$

Substituting $\frac{80}{100} = \frac{16}{x_2}$

Solving $x_2 = 20$

Example 2 If x varies inversely as y and $x = 8$ when $y = 2$, find x when $y = 4$

Solution 1 $xy = k$

Then $8 \cdot 2 = k$ and $k = 16$

When $y = 4$, $x \cdot 4 = 16$, and $x = 4$

Solution 2 Since x and y vary inversely,

$$\frac{x_1}{x_2} = \frac{y_2}{y_1}$$

Notice the arrangement of the subscripts in the proportion

$$\frac{8}{x_2} = \frac{4}{2}$$

Solving $x_2 = 4$

EXERCISES

[A]

1. x varies directly as y and $x = 153$ when $y = 9$ Find x when $y = 13$

2. S varies directly as P and $S = 198$ when $P = 22$ Find S when $P = 182$

3. Find x when $y = 32$ given that x varies inversely as y and that $x = 132$ when $y = 4$

4. x varies inversely as y and $x = 20$ when $y = 30$ Find y when $x = 25$

5. Two quantities vary inversely When the value of one of the quantities is 18 the value of the other one is $2\frac{1}{3}$ Find the value of the second quantity when the value of the first one is 25

6. The formula for discount is $d = rl$, r being the rate and l the list price

   a. If r is a constant how do d and l vary?

   b. If l is a constant how do d and r vary?

   c. If d is a constant how do r and l vary?

7. The cost of a plane trip varies directly as the distance Write a formula expressing this fact

8. The intensity of illumination from a point source of light varies inversely as the square of the distance from the source Write a proportion expressing this relation if I_1 and I_2 are intensities of illumination at distances R_1 and R_2 respectively

9. Express $\frac{V_1}{V_2} = \frac{t_1}{t_2}$ as a variation

10. Express $p = \frac{k}{v}$ as a proportion

11. The force F of attraction between two bodies varies jointly as the weights of the two bodies and inversely as the square of the distance between them Express this fact as a variation using k as a constant

Checking Your Understanding of Chapter 7

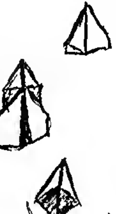
To be sure that you understand and can use the ideas presented in Chapter 7, check to see that you

1 Know the meaning of the word *ratio* and can express a ratio in several ways (p 189)

2. Know the meaning of the word *proportion*, can write a proportion in two ways, and can identify the extremes, the means, and the first, second, third, and fourth terms of a proportion (pp 190–191)

3 Know that in a proportion the product of the extremes equals the product of the means (p 191)

4. Can solve problems involving proportion (p 192) This includes ability to find the mean proportional between two numbers, the third proportional to two numbers, and the fourth proportional to three numbers



5 Understand the meaning of direct variation (p 194), can express it algebraically in two ways, and recognize that a direct variation produces a straight-line graph

6. Understand the meaning of inverse variation (p 195), can express it algebraically in two ways (p 197), and recognize that an inverse variation produces a curved line graph (p 195)

7. Understand joint variation and can express it algebraically (p 196)

8 Can solve problems involving direct, inverse and joint variation (pp 196–198)

[A]

1. Define ratio, proportion, means of a proportion, extremes of a proportion

2 The rate of an automobile covering a distance of 100 miles varies inversely as the time Express the fact by a formula

3. Find the ratio of

a. 10 to 35   c 4 feet to 3 yards

b 45 to 30   d. 80 rods to a mile

4. Find the mean proportional between 4 and $6\frac{1}{4}$

5. Find the fourth proportional to 10, 15, and 20

6. If 8 pounds of meat can be bought for \$6.32, how many pounds can be bought for \$3.95?

7. The sides of one triangle are 18 inches, 20 inches, and 24 inches. Find the shortest side of a similar triangle if its longest side is 30 inches.

8. If the area of the smaller triangle in Exercise 7 is 176.2 square inches, what is the area of the larger?

9. How do x and y vary in the equation $x = \frac{100}{y}$?

10. p varies directly as q. When $q = 31.2$, $p = 20.8$. Find p when $q = 15.3$.

11. Change the variation $V = \frac{k}{r}$ into a proportion.

12. Graph the equation $xy = 10$.

13. If y varies inversely as x and $y = 50$ when $x = 10$, what is the value of y when $x = 20$?

14. Find the third proportional to 3 and 9.

CHAPTER TESTS

[Test A]

1. Two quantities vary inversely. If the value of the first is 15 when the value of the second is 18, find the value of the second quantity when the value of the first is 10.

2. Find the fourth proportional to 7, 3.5, and 6.

3. Find the mean proportional between 5 and 45.

4. Divide 360 into 3 parts which shall have the ratio 2 : 3 : 5.

5. The means of a proportion are 8 and 18. If one of the extremes is 2, what is the other?

6. The number of feet a body falls varies directly as the square of the number of seconds during which it falls. If a body falls 16.1 feet during the first second, how far will it fall in 4 seconds?

7. Write in algebraic language: At a given temperature the electrical resistance R of a piece of copper wire varies directly as its length L and inversely as the square of its diameter D. Use k as a constant determined by the material and the temperature.

[Test 8]

1. The number of revolutions of two pulleys is inversely proportional to their diameters. If a 24-inch pulley making 400 revolutions per minute is belted to an 8 inch pulley, find the number of revolutions per minute of the smaller pulley.

2.

x	5	7	9
y	15	21	27

Tell whether the variation shown in the table is direct or inverse. Write the equation expressing the relationship.

3. The lengths of a pair of corresponding sides of two similar triangles are 6 inches and 9 inches respectively. If the area of the smaller triangle is 12 square inches, what is the area of the larger?

4. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. How is V affected when r is increased by one half of itself?

5. The formula $E = mv^2$ tells how to find the energy, E, when the mass, m, and the velocity, v, are known. If m represents the weight of a ball bat and v the velocity of the ball struck by the bat, which sends the ball farther: to use a bat twice as heavy or to double its velocity?

6. If u varies jointly as x and y and inversely as the square of z, and if $u = 280$ when $x = 30$, $y = 12$, and $z = 3$, find u when $x = 20$, $y = 10$, and $z = 2$.

$\frac{xy}{wz^2}$

$\frac{wz^2}{y} = 2$

$26 = 10$

CHAPTER 8

Exponents, Radicals, and Imaginaries

In this chapter you will learn more about exponents and radicals and find a new kind of number

Laws of Exponents [A]

So far in our work we have used only positive integral exponents In many applications of mathematics it is necessary to use fractional negative and zero exponents

We shall define these exponents so that they will conform to the laws for positive integral exponents For easy reference the laws for positive integral exponents are stated here

1 Law of Multiplication

$$x^m \cdot x^n = x^{m+n}$$

PROOF $\quad x^m = x \cdot x \cdot x \cdot x \quad$ to m factors

and $\quad x^n = x \cdot x \cdot x \cdot x \quad$ to n factors

$$x^m \cdot x^n = x \cdot x \cdot x \cdot x \cdot x \quad \text{to } (m+n) \text{ factors} = x^{m+n}$$

Examples $x^3 \cdot x^2 = x^5 \quad (x+y)^2(x+y) = (x+y)^3$

2 Law of Division

$$x^m \div x^n = x^{m-n} \quad (m > n)$$

PROOF $\quad x^m = x \cdot x \cdot x \cdot x \quad$ to m factors

and $\quad x^n = x \cdot x \cdot x \cdot x \quad$ to n factors

$$\frac{x^m}{x^n} = \frac{x \cdot x \cdot x \cdot x \quad \text{to } m \text{ factors}}{x \cdot x \cdot x \quad \text{to } n \text{ factors}}$$

$$= x \cdot x \cdot x \quad \text{to } (m-n) \text{ factors}$$

$$= x^{m-n}$$

Examples $\frac{x^5}{x^2} = x^3 \quad \frac{x^n}{x^4} = x^{n-4} \quad \frac{(x-y)^3}{x-y} = (x-y)^2$

3 Law of Power of a Power

$$(x^m)^n = x^{mn}$$

PROOF $\quad (x^m)^n = x^m \cdot x^m \cdot x^m \quad$ to n factors

$$= x^{m+m+m} \quad \text{to } n \text{ terms}$$

$$= x^{mn}$$

Examples $(x^3)^2 = x^6 \quad (2x^2)^3 = 2^3 \cdot x^6 = 8x^6, \quad (2a^3)^n = 2^n a^{3n}$

4 Law of Powers of Products

$$(xy)^n = x^n y^n$$

PROOF

$$(xy)^n = xy \cdot xy \cdot xy \cdot xy \quad \text{to } n \text{ factors}$$

$$= x \cdot x \cdot x \cdot x \quad \text{to } n \text{ factors} \cdot y \cdot y \cdot y \cdot y \quad \text{to } n \text{ factors}$$

$$= x^n y^n$$

5 Law of Powers of Quotients

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

PROOF

$$\left(\frac{x}{y}\right)^n = \frac{x}{y}\ \frac{x}{y}\ \frac{x}{y}\ \frac{x}{y} \quad \text{to } n \text{ factors}$$

$$= \frac{x\ x\ x\ x \quad \text{to } n \text{ factors}}{y\ y\ y\ y \quad \text{to } n \text{ factors}}$$

$$= \frac{x^n}{y^n}$$

EXERCISES [A]

Perform the indicated operations

1. $a^6\ a^2$
2. $a^6 + a^2$
3. $(a^6)^2$
4. $m^7 + m^4$
5. $3x^2\ 4x^3$
6. $a^n\ a^3$
7. $(\frac{2}{5})^3$
8. $\dfrac{(4ab)^2}{(2c)^3}$
9. $\left(-\dfrac{3ab}{2a^2c}\right)^2$
10. $(2)^3$
11. $(-2)^5$
12. -2^4
13. $(-2)^4$
14. $x^2\ y^3\ x\ y$
15. $(m^2)^3(3m)^2$
16. $\dfrac{b^{5n}}{b^{2n}}$
17. $\dfrac{a^{x+5}}{a^{3-x}}$
18. $\dfrac{a^x}{a^{2x}}$
19. $(a^2b^3)^3$
20. $(a^2b^3)^n$
21. $(2x^2)^n$
22. $(a^nb^n)^n$
23. $(x^ny^{2n})^4$
24. $(3a^{2n})^3$
25. $\dfrac{a^{x+4}b}{a^xb}$
26. $\dfrac{a^{3m}b^n}{a^mb^2}$
27. $\left(\dfrac{x^{n+3}}{y}\right)^2$

Roots and Radicals [A]

You are familiar with the meaning of 3^2, $(-3)^2$, and the square root of 9 Both 3 and -3 are the square roots of 9, since $(3)(3) = 9$ and $(-3)(-3) = 9$ A square root of a number is one of the two equal factors of the number

Any number has three cube roots A cube root of a number is any one of the three equal factors of the number One cube root of 27 is 3 because $3 \times 3 \times 3 = 27$ The two other cube roots of 27 are complex numbers, which we shall study later in this chapter One cube root of -27 is -3 because $(-3)(-3)(-3) = -27$ The two other cube roots of -27 are complex numbers Any number, excluding zero, has two square roots, three cube roots, and n nth roots

A radical is any indicated root of a number. Thus, $\sqrt{4}$, $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt[3]{-8}$, and $\sqrt[n]{a}$ are radicals. The numbers 4, 9, 8, -8, and a in the preceding sentence are called radicands. The symbol $\sqrt{\ }$ is called the radical sign. As in the examples above, it is usually combined with the bar and written in the form $\sqrt{n}$. The index, or order, of the radical is the small figure or letter written above the radical sign to indicate which root is to be taken. Thus $\sqrt[2]{25}$ means the square root of 25, $\sqrt[3]{8}$ means the cube root of 8, and $\sqrt[n]{a}$ means the nth root of a.

When the square root of a number is to be taken, the index 2 is usually omitted. For example, $\sqrt{4}$ means the square root of 4. Since there are two square roots of 4, you may ask "Which of the two square roots of 4 is meant by the expression $\sqrt{4}$?" Mathematicians have agreed that $\sqrt{4}$ shall mean $+2$. They call $+2$ the principal square root of 4. We indicate the negative square root of 4, which is -2, by placing the minus sign before the radical, thus $-\sqrt{4}$. Likewise $\sqrt{9}$ means $+3$ and $-\sqrt{9}$ means -3. We indicate both square roots of a number by placing the $+$ and $-$ signs before the radical. Thus $\pm\sqrt{9} = \pm 3$.

The principal cube root of 8 is $+2$ and it is indicated by $\sqrt[3]{8}$. The principal cube root of -8 is -2 and it is indicated by $\sqrt[3]{-8}$.

In general, we use the symbol $\sqrt[n]{a}$ to indicate the principal nth root of a when a has a real nth root. You should remember that a radical which has either a plus sign or no sign before it indicates the principal root.

The following statements, in which n is an integer, refer to principal roots.

(1) $\sqrt[n]{a}$ is positive when a is positive.

(2) $\sqrt[n]{a}$ is negative when a is negative and n is odd.

(3) $\sqrt[n]{a}$ is imaginary when a is negative and n is even. (An imaginary number is an even root of a negative number. You will study these a little later.) In this case, we do not speak of $\sqrt[n]{a}$ as being a principal root.

We often have need to simplify radicals of the form $\sqrt[n]{a^n}$. When n is odd, $\sqrt[n]{a^n} = a$ for both positive and negative values of a.

Examples: If $n = 3$, and $a = 2$, $\sqrt[n]{a^n} = \sqrt[3]{8} = +2 = a$
If $n = 3$, and $a = -2$, $\sqrt[n]{a^n} = \sqrt[3]{-8} = -2 = a$

When n is even, we cannot tell whether $\sqrt[n]{a^n}$ equals $+a$ or $-a$ until we know the sign of a. It can be shown that

(1) $\sqrt[n]{a^n} = +a$ when n is even and a is positive
(2) $\sqrt[n]{a^n} = -a$ when n is even and a is negative

Examples If $n = 2$ and $a = +3$, $\sqrt[n]{a^n} = \sqrt{9} = +3 = a$
If $n = 2$ and $a = -3$, $\sqrt[n]{a^n} = \sqrt{9} = +3 = -a$

NOTE To avoid this double meaning of the principal value and to exclude imaginary numbers, we shall assume (unless stated otherwise) the following

If the index of a radical is an even integer, all the letters in the radicand except exponents will denote positive numbers and the radicand will be positive. With this agreement, $\sqrt[n]{a^n} = +a$ for all cases in this book

Examples $\sqrt{a^2} = a$ $\qquad \sqrt[3]{-8c^3} = -2c$
$\sqrt{4x^2} = 2x$ $\qquad \sqrt[3]{27y^6} = 3y^2$
$\sqrt[4]{256a^4} = 4a$ $\qquad \sqrt[5]{+32x^{10}} = 2x^2$

[A]

ORAL EXERCISES

Find the indicated roots

1. $\sqrt{1}$	6. $-\sqrt{9}$	11. $\sqrt[3]{-64}$	16. $\sqrt{(a+b)^2}$
2. $\sqrt{9}$	7. $-\sqrt{64}$	12. $\sqrt[3]{125}$	17. $\sqrt{(x-1)^2}$
3. $\sqrt{25}$	8. $\sqrt{x^2}$	13. $\sqrt[6]{64a^{12}}$	18. $\sqrt[3]{(a-b)^3}$
4. $\sqrt{49}$	9. $\sqrt{y^6}$	14. $\sqrt[3]{-27y^3}$	19. $-\sqrt{1.44s^2}$
5. $\sqrt{81}$	10. $\sqrt{49x^4}$	15. $\sqrt[3]{x^3y^9}$	20. $-\sqrt{+25a^4b^6}$

Evolution of Radicals[A]

In computations we need to evaluate radicals of the second order much more frequently than radicals of a higher order. All roots of numbers may be found by logarithms, which will be explained in the next chapter. It is usually quicker to find the square root of a number either by the arithmetic method or by using a table of square roots. We know that the square roots of most numbers are only approximate values. Thus $\sqrt{2} = 1.4$, or 1.41, or 1.414, or 1.4142, depending on the degree of accuracy we wish to use

You are familiar with the arithmetic method of finding square root from previous courses in mathematics. We shall give two examples to assist you in recalling the process

Example 1 Evaluate $\sqrt{106929}$

Solution 1 Divide the radicand into groups of two digits each starting at the decimal point

2 Determine the largest perfect square less than 10 The largest perfect square is 9 or $3^2 = 9$ Place the 3 above 10 as the first digit in the root subtract 9 from 10 and bring down the next group

3 Obtain a trial divisor by multiplying 3 the root already found by 2 and write the 6 to the left of 169

4 6 will divide into 16 two times Place 2 after 6 and above the group 69

5 Multiply 62 by 2 and subtract the product 124 from 169 Bring down the next group

6 As in step 3 obtain a trial divisor by multiplying 32 the root already found by 2 and write the 64 to the left of 4529

7 As in step 4 64 will divide into 452 seven times Place 7 after 64 and above the group 29

8 Multiply 647 by 7 and subtract the product 4529 from 4529 The remainder is zero so $\sqrt{106929} = 327$

```
       3  2  7
      10 69 29
       9
62   | 1 69
     | 1 24
647  |   45 29
     |   45 29
```

Example 2 Evaluate $\sqrt{4026}$ to the nearest tenth

Solution The computation is shown at the right You should follow the solution through referring to Example 1 if there is any part you do not recall Notice that the square root is found to two decimal places so that the result can be given to the nearest tenth

The positive square root of 4026 to the nearest tenth is 63 5

```
         6  3 . 4  5 +
        40 26 . 00 00
        36
123    | 4 26
       | 3 69
1264   |   57 00
       |   50 56
12685  |    6 44 00
       |    6 34 25
```

(A) **EXERCISES**

Find the positive square root of

1 45369	3 10816	5 27225	7 75625
2 5184	4 1156	6 43264	8 219961

Find the positive square root to the nearest tenth

9 505	11 10 43	13 4307	15 1 4
10 683	12 62 4	14 2246	16 7 3

Fractional Exponents[A]

If we apply the first law of exponents to $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}}$, we have $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2$. But $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, hence $a^{\frac{2}{3}}$ has the same meaning as $\sqrt[3]{a^2}$, the principal cube root of a^2. Therefore, if we wish the law of exponents to hold for fractional exponents, we must define $a^{\frac{2}{3}}$ as equal to $\sqrt[3]{a^2}$ or, in general,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

the principal nth root of a^m.

> The numerator of a fractional exponent denotes the power of the base.
> The denominator of a fractional exponent denotes the root of the base.

Example 1. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

Example 2. $125^{\frac{5}{3}} = \left(\sqrt[3]{125}\right)^5 = 5^5 = 3125$

EXERCISES (A)

Express as radicals

1. $a^{\frac{1}{3}}, a^{\frac{3}{4}}, a^{\frac{1}{2}}, a^{\frac{2}{5}}, a^{\frac{3}{2}}$
2. $x^{\frac{2}{3}}, x^{\frac{3}{5}}, x^{\frac{1}{4}}, x^{\frac{3}{4}}, x^{\frac{5}{2}}$
3. $(x-y)^{\frac{1}{2}}, (x-y)^{\frac{2}{3}}, (x-y)^{\frac{3}{4}}$
4. $2a^{\frac{1}{2}}, 3a^{\frac{2}{3}}, 5a^{\frac{1}{4}}, 7a^{\frac{3}{5}}$

Express without radical signs

5. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[3]{a^2}$, $\sqrt[5]{a^3}$, $\sqrt[4]{a^3}$, $3\sqrt{a}$, $5\sqrt[3]{a}$
6. $\sqrt{x+y}$, $\sqrt[3]{x+y}$, $\sqrt[3]{(x+y)^2}$, $3\sqrt{(x+y)}$

Evaluate

7. $25^{\frac{1}{2}}$	11. $125^{\frac{2}{3}}$	15. $(\frac{27}{64})^{\frac{1}{3}}$	19. $(\frac{4}{9})^{\frac{3}{2}}$
8. $4^{\frac{3}{2}}$	12. $(\frac{1}{16})^{\frac{1}{4}}$	16. $9^{\frac{3}{2}}$	20. $(0.027)^{\frac{1}{3}}$
9. $27^{\frac{2}{3}}$	13. $(16)^{\frac{3}{4}}$	17. $144^{\frac{1}{2}}$	21. $(0.125)^{\frac{2}{3}}$
10. $81^{\frac{1}{2}}$	14. $(\frac{8}{27})^{\frac{2}{3}}$	18. $32^{\frac{3}{5}}$	22. $(1.21)^{\frac{1}{2}}$

Multiply

23. $a^{\frac{1}{2}} \cdot a^{\frac{3}{4}}$	25. $b \cdot b^{\frac{1}{2}}$	27. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$	29. $a \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{5}}$
24. $b^{\frac{2}{3}} \cdot b^{\frac{5}{3}}$	26. $c^{\frac{1}{2}} \cdot c^{\frac{3}{4}}$	28. $y^{\frac{1}{2}} \cdot y^{\frac{2}{3}}$	30. $n^{\frac{1}{3}} \cdot n^{\frac{1}{2}} \cdot n^{\frac{1}{4}}$

Divide

31 $\dfrac{a^{\frac{3}{4}}}{a^{\frac{1}{4}}}$	33 $\dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}$	35 $\dfrac{2y^{\frac{3}{4}}}{y^{\frac{1}{2}}}$	37 $\dfrac{x^{0.5}}{x^{0.25}}$
32 $\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}}$	34 $\dfrac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$	36 $\dfrac{3m^{\frac{4}{5}}}{m^{\frac{1}{3}}}$	38 $\dfrac{a^{2.718}}{a}$

Raise to the indicated powers

39 $(a^{\frac{1}{3}})^3$	41 $(2^{\frac{1}{3}})^{\frac{3}{2}}$	43 $(a^{\frac{3}{4}})^{\frac{1}{2}}$	45 $(a^2b^3)^{\frac{1}{6}}$
40 $\left(\dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^3$	42 $\left(\dfrac{x^2}{y^2}\right)^{\frac{1}{3}}$	44 $\left(\dfrac{x^{\frac{1}{2}}}{y^2}\right)^2$	46 $\left(\dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{2}}}\right)^6$

Zero and Negative Exponents [A]

In order that the law of exponents $x^m \cdot x^n = x^{m+n}$ may hold for a zero exponent, it is necessary to define x^0 so that $x^0 \cdot x^n = x^{0+n} = x^n$. Since $1 \cdot x^n = x^n$, we define $x^0 = 1$ if $x \neq 0$.

The zero power of any number except 0 is 1

We do not define 0^0, as it has no meaning.

Example $\dfrac{a^0}{3x^0} = \dfrac{1}{3 \cdot 1} = \dfrac{1}{3}$

If the law of exponents is to hold for negative exponents, we must have

$$x^m \cdot x^{-m} = x^0 = 1$$

D_{x^m}

$$x^{-m} = \frac{1}{x^m}$$

Hence we define x^{-m} as $\dfrac{1}{x^m}$, where $x \neq 0$.

In like manner, if $x^m \cdot x^{-m} = 1$,

$D_{x^{-m}}$

$$x^m = \frac{1}{x^{-m}}$$

Any factor of either term of a fraction may be transferred to the other term provided the sign of its exponent is changed

In using this rule we must be careful to distinguish factors from terms.

Examples $\frac{3}{2x^{-1}} = \frac{3x}{2}$, $\frac{1}{2(x-y)^{\ \frac{1}{2}}} = \frac{(x-y)^{\frac{1}{2}}}{2}$

EXERCISES

[A]

Express with positive exponents

1 $\frac{a^2b^{\ 1}}{c}$

2 a^{-4}

3 $\frac{a^{\ 2}}{x^{\ 3}}$

4 $\frac{2a^{\ 3}}{b}$

5 $a^{\ 1} + b^{-1}$

6 $\frac{a^{\ 1}b^{\ 2}}{c^{\ 3}}$

7 $x^{\ 2}y^3$

8 $\frac{a^{\ 2}}{3}$

9 $\frac{2a^{\ 3}}{5b^{\ 2}}$

10 $xy^{-1} + x^{\ 1}y$

11 $\frac{2}{a^{-\frac{1}{2}}}$

12 $5a^{-5}$

13 $\frac{b^{\ \frac{3}{4}}}{b}$

14 $\frac{a^{\frac{1}{2}}}{2a^{-\frac{1}{4}}}$

15 $3(a+b)^{-2}$

State the value of

16 3^0

17 $5^0\ 2$

18 $5a^0$

19 $(5a)^0$

20 $\frac{4}{2x^0}$

21 $\frac{1+a^0}{2}$

22 $2^{\ 1}$

23 $4^{\ 2}$

24 $4^{\ \frac{1}{2}}$

25 $16^{-\frac{3}{4}}$

26 $5^{\ 2}$

27 $49^{\frac{1}{2}}$

28 $\frac{4}{3^{-1}}$

29 $7^0\ 49^{\ \frac{1}{2}}$

30 $3^{-2}\ 27^{\frac{1}{3}}$

31 $64\ 4^{\ 2}$

32 $32\ 2^{\ 4}$

33 $4^{\frac{1}{2}}(\frac{1}{2})^{\ 4}$

34 $3\ 10^{-4}$

35 $(\frac{1}{25})^{-\frac{1}{2}}$

36 $(\frac{2}{3})^{-1}$

37 $5 + 8^0$

38 $5\ 8^0$

39 $(2^0)^{-4}$

40 $(-4)^{-2}$

41 $(50)^0\ (25)^{-\frac{1}{2}}$

42 $8^{-\frac{2}{3}}$

43 $\frac{1}{(2+b^0)^{-2}}$

44 $\frac{1}{2^0+b^0}$

Find the simplest value in each example

45 $(m^2)^{\ 2}$

46 $\sqrt{10^2}$

47 $9^{\frac{1}{2}}\ 8^{\frac{1}{3}}$

48 $(y^{-2})^2$

49 $c^{-a}\ c^{+a}$

50 $x^{-3} + x^{3}$

51 $(\ 04)^{\frac{1}{2}}$

52 $x^2 + x^{-2}$

53 $(10^{\frac{1}{2}})^4$

54 $\sqrt{x^{\ 4}}$

55 $x^2 + x^{\frac{1}{2}}$

56 $10^{\frac{1}{2}} + 10$

57 $(\ 01)^{\frac{1}{2}}$

58 $64^{-\frac{1}{2}}$

59 $25 + 5^2$

60 $b^{\frac{3}{4}} \div b^{-\frac{1}{4}}$	64 $x^3 \times x^{\frac{1}{2}}$	68 $(x^{4a})^{\frac{1}{4}}$
61 $x^7 \div x^{-7}$	65 $(m^{-2})^3$	69 $(-125a^3)^{\frac{2}{3}}$
62 $7x^0$	66 $(x^6)^{\frac{2}{3}}$	70 $(-64x^6)^{\frac{2}{3}}$
63 $x^{-4} \times x^2y$	67 $10^{-3} \times 10^4$	71 $10{,}000 \times 10^{-4}$

Find the following special products

72 $a(a - 1 + a^{-1})$	76 $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$
73 $b^{\frac{1}{2}}(b^2 + b^{\frac{3}{2}} - b)$	77 $(x^2 - x^{-2})^2$
74 $x^{\frac{1}{3}}(x^{\frac{2}{3}} + 1 + x^{-\frac{1}{3}})$	78 $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$
75. $m^{\frac{1}{2}}(2m^{\frac{1}{2}} - 3 + 5m^{-\frac{1}{2}})$	79 $(x^{\frac{1}{2}} + 3)(x^{\frac{1}{2}} - 5)$

Multiply

80 $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$

81 $(a^{-1} - 2)(2a^{-2} - 5a^{-1} + 7)$

82 $(m^{-1} - n^{-1})(m^{-2} + m^{-1}n^{-1} + n^{-2})$

Divide

83. $(x - x^{\frac{1}{2}} + x^{\frac{1}{4}})$ by $x^{\frac{1}{4}}$	86 $(m - n)$ by $(m^{\frac{1}{2}} + n^{\frac{1}{2}})$
84 $(y^2 - y + y^{-2})$ by y^{-3}	87 $(a + b)$ by $(a^{\frac{1}{3}} + b^{\frac{1}{3}})$
85. $(c^{\frac{4}{3}} + c^{\frac{2}{3}} + 1)$ by $c^{-\frac{2}{3}}$	88 $(x + 1)$ by $(x^{\frac{1}{3}} + 1)$

89 $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}})$ by $(x + y)$

90 $(e^{2x} + 2 + e^{-2x})$ by $(e^x + e^{-x})$

Factor

91 $x^{-2} - y^{-2}$	94 $e^{2x} - 2 + e^{-2x}$
92 $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y$	95. $6 - x^{-1} - x^{-2}$
93 $16 - a^{-2}$	96 $6y^{-4} + 13y^{-2} + 5$

Standard or Scientific Notation of Numbers [A]

In science and engineering very large or very small numbers like 93,000,000 and 0.0000000666 are written as the product of one number having one digit to the left of the decimal point and another number which is a power of 10. Thus

$$93{,}000{,}000 = 9.3 \times 10^7,$$
$$0.0000000666 = 6.66 \times 10^{-8}$$

The numbers 9.3×10^7 and 6.66×10^{-8} are said to be written in <u>scientific</u> or <u>standard notation</u>. Why do you think scientists use standard notation in their work?

Since multiplying a number by 10 moves the decimal point one place to the right and dividing a number by 10 moves the decimal point one place to the left, the exponent of 10 can be written by inspection.

Example 1 $0.00000340 = 3.40 \times 10^{-6}$

Example 2 $0.005305 = 5.305 \times 10^{-3}$

Example 3 $5{,}240{,}000 = 5.24 \times 10^6$

Example 4 $932 = 9.32 \times 100 = 9.32 \times 10^2$

Example 5 $0.063 = 6.3 \times \frac{1}{100} = 6.3 \times 10^{-2}$

Example 6 $3.86 = 3.86 \times 1 = 3.86 \times 10^0$

EXERCISES (A)

Express in standard notation.

1. 37,000,000,000,000
2. 5,000,000,000
3. 707,000,000,000,000
4. 27,510,000
5. 0.0000014
6. 0.00000000030
7. 0.00000283
8. 0.000000702

Write without exponents.

9. 1.770×10^6
10. 1.25×10^7
11. 3.2×10^8
12. 7.2×10^{-12}
13. 3.6×10^{-5}
14. 1.663×10^{-24}

Express in standard notation.

15. 6000
16. 720

F1.05

45 (m300

46 $\sqrt{10}$

20. 47.6
21. 876
22. 0.43
23. 0.070
24. 0.0083
25. 933.4
26. 0.924
27. 0.042
28. 100
29. 9500

47. $9^{\frac{1}{2}}$ $8^{\frac{1}{3}}$ earth is about 93,000,000 miles from the sun. Ex-

tance in standard notation.

48 $(y^{-2})^2$

49 c^{-a} c^{+a} ameter of an average red corpuscle is 0.0008 cen-

te the size of the corpuscle in standard notation.

32. The *radius of an electron is about* (2×10^{-13}) centimeters. Express the radius as a decimal fraction without the use of an exponent.

33. The wave length of red light is (7.60×10^{-5}) centimeters. Express this wave length as a decimal fraction without the use of an exponent.

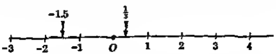
34. In one summer a pair of house flies can become parents and ancestors of 1.91×10^{20} flies. Express this number as a decimal fraction without the use of an exponent.

Rational Numbers [A]

As was said in Chapter 1, the first numbers used by man were integers (whole numbers), which were used in counting objects. Later, fractions were used to denote parts of objects.

Negative numbers were not used as such until the sixteenth century, although years before this time the Chinese had used red and black colors on computing rods to show opposite numbers. Integers and fractions, both positive and negative, make up the rational number system.

A rational number is a number which can be expressed in the fraction form $\frac{a}{b}$, in which a is a positive or negative integer (including zero) and b is a positive integer. Thus, 4, $\frac{1}{2}$, $\frac{1}{3}$, $7\frac{1}{2}$, -2, 4.25, and .007 are rational numbers. Since a *rational* number can be expressed as a ratio, you can easily remember its meaning. Can you explain why an integer such as 4 is a rational number?

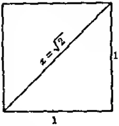


Any rational number can be represented by a point on the rational number scale. For each rational number there is a point on this scale. Do you think the converse of this statement is true?

Need for Irrational Numbers [A]

The early Greeks were aware of numbers which are not rational. Let us see the need of such numbers. If each side of a square is one inch long, we know by the Pythagorean theorem that the length x

of the diagonal is given by the equation $x^2 = 1^2 + 1^2$, or $x^2 = 2$ This means that the square of the diagonal equals 2 and that x equals the square root of 2 Since the diagonal has a definite length the $\sqrt{2}$ must be a definite number No doubt you have found its value to three decimal places to be 1 414 Yet 1 414 is not the exact value of $\sqrt{2}$ It can be shown that $\sqrt{2}$ is not a rational number For those students who are interested the proof is given in the next section



Proof that $\sqrt{2}$ Is Not a Rational Number[C]

We shall now use the indirect method of proof to show that $\sqrt{2}$ cannot be expressed as the quotient of two integers

To prove $\sqrt{2}$ is not a rational number

PROOF 1 $\sqrt{2}$ is rational or it is not rational

2 Since a rational number can be expressed as a fraction, suppose that $\sqrt{2} = \frac{a}{b}$ where $\frac{a}{b}$ is reduced to lowest terms

3 If $\sqrt{2} = \frac{a}{b}$, $2 = \frac{a^2}{b^2}$, and $a^2 = 2\,b^2$

4 Since 2 is a factor of $2\,b^2$, it is a factor of a^2

5 If 2 is a factor of a^2, it is a factor of a

6 If 2 is a factor of a, it is twice a factor of a^2

7 If 2 is twice a factor of a^2, it is twice a factor of $2\,b^2$

8 If 2 is twice a factor of $2\,b^2$, it is a factor of b^2

9 If 2 is a factor of b^2, it is a factor of b

10 From (5) and (9), 2 is a factor of a and b, and $\frac{a}{b}$ is not in lowest terms, which is contrary to hypothesis

11 From (1) and (10), $\sqrt{2}$ is not rational

Irrational Numbers; Surds[A]

The number $\sqrt{2}$ lies between all positive rational numbers whose squares are less than 2 and all positive rational numbers whose squares are greater than 2 The number $\sqrt{2}$ is called an irrational number

Although we have shown the existence of only one number that is not rational, there is an infinite number of such numbers

An irrational number is a real number that is not rational (We shall study numbers that are not real later in this chapter) Other examples of irrational numbers are $\sqrt{5}$ $\sqrt{15}$ $\sqrt{7}$ and π

An irrational number that is the indicated root of a rational number is called a surd A surd is an indicated root which can be found only approximately like $\sqrt{2}$ $\sqrt{3}$ and $\sqrt[3]{20}$

A surd of the second order like $\sqrt{2}$ and $\sqrt{15}$ is a quadratic surd

When all the rational and irrational numbers are plotted on the real number scale the following statement is true

For each number there is a point on the scale and for each point on the scale there is a real number

Properties of Radicals [A]

When working with radicals you should be governed by the following properties which are expressed by formulas In these formulas a and b are real numbers and m and n are integers

Property I $(\sqrt[n]{a})^n = a$ (Definition of nth root)

Examples $(\sqrt{3})^2 = 3$ $(\sqrt[3]{8})^3 = 8$

Property II $\sqrt[n]{a^n} = a$ (See pages 205 206)

Property III $\sqrt[n]{a^n} = (\sqrt[n]{a})^n$

Property IV $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

Examples $\sqrt{36} = \sqrt{4}\,\sqrt{9}$ $\sqrt[3]{-216} = \sqrt[3]{-8}\,\sqrt[3]{27}$

Property V $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, when $b \neq 0$

Examples $\sqrt{\dfrac{4}{25}} = \dfrac{\sqrt{4}}{\sqrt{25}}$ $\sqrt[3]{\dfrac{27}{64}} = \dfrac{\sqrt[3]{27}}{\sqrt[3]{64}}$

Property VI $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Examples $\sqrt{\sqrt[3]{64}} = \sqrt[6]{64}$, $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125}$

Property VII $\sqrt[mp]{a^{np}} = \sqrt[m]{a^n}$

NOTE When applying these formulas you should remember the note on page 206

Simplifying Radicals [A]

Some radicals can be simplified by finding their indicated roots while others can be simplified by changing them into other radicals Mathematicians agree that a radical is in its simplest form when

1 The radicand does not contain a factor whose indicated root can be taken

2 The radicand does not contain a fraction

3 The index of the radical is the smallest possible integer

Simplifying Radicals with Integral Radicands [A]

When simplifying radicals you should remember to make the radicand the smallest possible whole number

Example 1 $\sqrt{20} = \sqrt{4}\ \sqrt{5} = 2\sqrt{5}$

Example 2 $\sqrt[3]{24\,x^4} = \sqrt[3]{8\,x^3}\ \sqrt[3]{3\,x} = 2\,x\sqrt[3]{3\,x}$

Example 3 $5\sqrt{72\,x} = 5\sqrt{36}\ \sqrt{2\,x} = 30\sqrt{2\,x}$

Rule for Simplifying Radicals Having Integral Radicands

Factor the radicand into two factors
one of which is the greatest perfect power
of the same degree as the radical
Take the indicated root of this factor
which then becomes a coefficient of the resulting surd

EXERCISES [A]

Simplify

1 $\sqrt{8}$	11. $\sqrt{48\,xy^2}$	21 $2\sqrt[4]{81\,t^3}$
2 $\sqrt{12}$	12 $\sqrt{121\,m^3}$	22 $3\sqrt{250\,x^3}$
3 $\sqrt{24}$	13 $\sqrt[3]{64\,x^2}$	23 $5\sqrt[3]{8\,a^4}$
4 $\sqrt{18}$	14 $\sqrt[3]{250\,x^4}$	24 $a\sqrt{169\,a}$
5 $\sqrt{45}$	15 $\sqrt{98\,a^4b^3}$	25 $5\,x\sqrt[3]{16\,x^4}$
6 $\sqrt[3]{24}$	16 $\sqrt{200\,x^3y^2}$	26 $12\sqrt{448\,y}$
7 $\sqrt{50\,a}$	17 $\sqrt[3]{40\,x^4y}$	27. $\sqrt[5]{243\,a^6}$
8 $\sqrt[3]{56}$	18 $\sqrt[3]{64}$	28 $c\sqrt[3]{2\,b^5c}$
9 $\sqrt[3]{54\,x^4}$	19. $\sqrt[4]{32\,a^5}$	29 $b\sqrt{0.09\,a^3b}$
10 $\sqrt{28\,x^2}$	20 $\sqrt[3]{-40\,a^3b^2}$	30 $4\,c\sqrt{0.16\,c^3}$

Simplifying Radicals with Fractional Radicands[A]

We simplify a radical with a fractional radicand by changing it to a radical in its simplest form having an integral radicand

Example 1. Simplify $\sqrt{\frac{1}{8}}$

Solution. Multiply both terms of $\frac{1}{8}$ by 2 to make the denominator a perfect square

Then $$\sqrt{\frac{1}{8}}=\sqrt{\frac{2}{16}}=\frac{\sqrt{2}}{\sqrt{16}}=\frac{\sqrt{2}}{4}$$

Example 2 Simplify $\sqrt{\frac{4}{3a}}$

Solution $$\sqrt{\frac{4}{3a}\cdot\frac{3a}{3a}}=\sqrt{\frac{12a}{9a^2}}=\frac{\sqrt{12a}}{3a}=\frac{2\sqrt{3a}}{3a}$$

Example 3 Simplify $\sqrt[3]{\frac{8}{3a^3}}$

Solution $$\sqrt[3]{\frac{8}{3a^3}\cdot\frac{9a}{9a}}=\frac{\sqrt[3]{72a}}{\sqrt[3]{27a^3}}=\frac{\sqrt[3]{72a}}{3a}=\frac{2\sqrt[3]{9a}}{3a}$$

Rule for Simplifying Radicals Having Fractional Radicands

Multiply both terms of the fraction
by the quantity which will make the denominator the least perfect power of the same degree as the index of the radical
Find the root of the denominator
and simplify the numerator

[A] **EXERCISES**

Simplify

1 $\sqrt{\frac{1}{2}}$

2 $\sqrt{\frac{2}{3}}$

3 $\sqrt{\frac{4}{5}}$

4. $\sqrt{\frac{8}{3}}$

5 $\sqrt{\frac{2x}{7}}$

6 $\sqrt[4]{\frac{3}{8x^3}}$

7 $\sqrt[3]{\frac{3}{4}}$

8 $\sqrt{\frac{5}{12}}$

9 $\sqrt[3]{\frac{2}{9}}$

10 $\sqrt{\frac{a}{b}}$

11 $\sqrt{\frac{4x^{2n}}{9y}}$

12 $\sqrt[3]{\frac{5m^3}{8n^2}}$

13 $\sqrt[3]{\frac{2x}{25}}$

14. $\sqrt[3]{\frac{ac}{b^2}}$

15. $\sqrt{\frac{3x}{11y}}$

16 $\sqrt{\frac{ab}{300}}$	22 $\sqrt{\frac{x^2-y^2}{x^2+y^2}}$	28 $\sqrt[3]{-\frac{3c^4}{8ab}}$
17 $\sqrt{\frac{1}{x-y}}$	23 $\frac{a}{b}\sqrt{\frac{3b^2}{8a^3}}$ $\frac{\sqrt{6a}}{4a}$	29 $\sqrt{x+\frac{y^2}{x}}$
18 $\sqrt{\frac{a-b}{a+b}}$	24 $\frac{2}{5}\sqrt[5]{\frac{9}{16}}$	30 $\sqrt{c^2-\left(\frac{c}{2}\right)^2}$
19 $\sqrt[3]{\frac{1}{(x+y)^2}}$	25 $\frac{1}{2}\sqrt{\frac{5y}{18x}}$ $\frac{\sqrt{10xy}}{12x}$	31 $\sqrt{a-\frac{4}{a}}$
20 $\sqrt[m]{\frac{a^{m+5}}{b^{m-2}}}$	26 $\sqrt[3]{-\frac{3x^4}{16y}}$	
21 $\sqrt[m]{\frac{a^{m+1}}{b^{m-1}}}$	27 $\sqrt{\frac{27a^5b^6}{32c^4}}$ $\frac{3a^2b^3\sqrt{6a}}{8c^2}$	32 $\sqrt{m^2-\frac{n^2}{4}}$

Simplifying Radicals by Lowering Their Orders [A]

You can learn the method of simplifying radicals by lowering their orders very quickly if you will use pencil and paper when studying the following examples

Example 1 Reduce $\sqrt[4]{4}$ to a lower order

Solution $\sqrt[4]{4}=\sqrt[4]{2^2}=2^{\frac{2}{4}}=2^{\frac{1}{2}}=\sqrt{2}$

Example 2 Reduce $\sqrt[6]{25x^4}$ to a lower order

Solution $\sqrt[6]{25x^4}=\sqrt[6]{5^2x^4}=5^{\frac{2}{6}}x^{\frac{4}{6}}=5^{\frac{1}{3}}x^{\frac{2}{3}}=\sqrt[3]{5x^2}$

Example 3 Reduce $\sqrt[9]{64}$ to a lower order

Solution $\sqrt[9]{64}=\sqrt[9]{2^6}=2^{\frac{6}{9}}=2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}$

We may also think of $\sqrt[9]{64}$ as $\sqrt[3]{\sqrt[3]{64}}=\sqrt[3]{4}$

EXERCISES [A]

Reduce the order of the following radicals

1 $\sqrt[4]{a^2}$	5 $\sqrt[4]{25a^2}$	9 $\sqrt[8]{81r^2s^4}$	13 $\sqrt[4]{16a^2b^2}$
2 $\sqrt[6]{x^3y^3}$	6 $\sqrt[9]{8x^3}$	10 $\sqrt[9]{125x^3}$	14 $\sqrt[10]{a^4b^2c^8}$
3 $\sqrt[4]{9}$	7 $\sqrt[10]{32}$	11 $\sqrt[4]{144a^2}$	15 $\sqrt[12]{64x^3y^6}$
4 $\sqrt[6]{27}$	8 $\sqrt[4]{16x^2y^2}$	12 $\sqrt[6]{125c^3}$	16 $\sqrt[4]{(a+b)^2}$

Addition and Subtraction of Radicals [A]

As in integral expressions we can add radicals if they are similar Two or more radicals are *similar* if their indices and radicands are

identical In all other cases the radicals are dissimilar Thus $\sqrt{3}$ and $5\sqrt{3}$ are similar, since they have the same index, 2, and the same radicand, 3 Also $\sqrt{12}$ and $\sqrt{48}$ are similar, since their radicands can be made identical, thus $\sqrt{12} = 2\sqrt{3}$, and $\sqrt{48} = 4\sqrt{3}$ The radicals $\sqrt{12}$ and $\sqrt{24}$ are dissimilar, since their radicands cannot be made identical, $\sqrt{12} = 2\sqrt{3}$, and $\sqrt{24} = 2\sqrt{6}$, and 3 is not identical with 6

Example 1. $2\sqrt{5} + 6\sqrt{5} - 3\sqrt{5} = 5\sqrt{5}$

Example 2 $3\sqrt{a} + 5\sqrt{b} - \sqrt{a} - 2\sqrt{b} = 2\sqrt{a} + 3\sqrt{b}$

Example 3. $\sqrt{75} + 3\sqrt{27} - 2\sqrt{48}$
$= 5\sqrt{3} + 9\sqrt{3} - 8\sqrt{3} = 6\sqrt{3}$

Example 4. $\sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{9}{8}} - \sqrt{\tfrac{1}{32}}$
$= \tfrac{1}{2}\sqrt{2} + \tfrac{3}{4}\sqrt{2} - \tfrac{1}{8}\sqrt{2} = \tfrac{9}{8}\sqrt{2}$

(A) **EXERCISES**

Simplify by combining similar radicals

1 $8\sqrt{3} + 2\sqrt{3} - 4\sqrt{3}$
2 $\sqrt{20} + \sqrt{45} - 4\sqrt{5}$
3 $\sqrt{12} + \sqrt{27} + \sqrt{48}$
4. $\sqrt{4x} - \sqrt{x} + \sqrt{25x}$
5 $\sqrt[3]{54} + \sqrt[3]{128} - 3\sqrt[3]{16}$
6. $9\sqrt[3]{250} + \sqrt[3]{16} - \sqrt[3]{\tfrac{1}{4}}$
7 $\sqrt{8} - \sqrt{72} + \sqrt{98}$
8 $\sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{1}{6}}$
9 $\sqrt{\tfrac{25}{3}} - 2\sqrt{\tfrac{16}{3}} + 2\sqrt{\tfrac{4}{3}}$
10 $3\sqrt{\tfrac{1}{12}} - \sqrt{\tfrac{1}{15}} + 5\sqrt{\tfrac{3}{5}}$
11 $\sqrt[3]{56} + \sqrt[3]{189} - \sqrt[3]{-7}$
12 $\sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{1}{6}} + \sqrt{\tfrac{1}{2}}$
13. $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} - \sqrt{\dfrac{1}{xy}}$
14 $\sqrt{(a+b)^2x} - \sqrt{(a-b)^2x}$
15. $\sqrt{(x+y)^3} + \sqrt{4(x+y)} - 3\sqrt{x+y}$
16. $\sqrt[3]{8ab^3 + 8b^4} + \sqrt[3]{27a^4 + 27a^3b} - \sqrt[3]{(a+b)^4}$

Multiplication of Radicals of the Same Order [A]

In multiplying radicals of the same order, write the product of the radicals under a single radical sign Then simplify the resulting radical

Example 1 $\sqrt{4} \cdot \sqrt{8} = \sqrt{32} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

Example 2 $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$

Example 3 $2\sqrt{6} \cdot 3\sqrt{2} = 6\sqrt{12} = 6\sqrt{4} \cdot \sqrt{3} = 6 \cdot 2\sqrt{3} = 12\sqrt{3}$

Example 4 $\sqrt[3]{5}\,\sqrt[3]{25} = \sqrt[3]{125} = 5$

Example 5 $\sqrt{2x}\,\sqrt{5x} = \sqrt{10x^2} = \sqrt{10}\,\sqrt{x^2} = \sqrt{10}\,x$

ORAL EXERCISES

[A]

State the products. odd

1 $\sqrt{5}\ \sqrt{2}$

2 $\sqrt{3}\ \sqrt{7}$

3 $\sqrt{2}\ \sqrt{18}$

4 $\sqrt{x}\ \sqrt{y}$

5 $\sqrt{2a}\ \sqrt{3b}$

6 $\sqrt[3]{2}\ \sqrt[3]{3}$

7 $\sqrt[3]{4x}\ \sqrt[3]{2x^2}$

8 $\sqrt{5x}\ \sqrt{5x}$

9 $\sqrt[3]{2}\ \sqrt[3]{4}$

[A]

Multiply as indicated

1 $\sqrt{3}\,\sqrt{18}$

2 $\sqrt{3}\,\sqrt{8}$

3 $\sqrt{8a}\,\sqrt{8a}$

4 $x\sqrt{a}\,\sqrt{b}$

5 $\sqrt{ab}\,\sqrt{ac}$

6 $\sqrt{\frac{1}{2}}\ \sqrt{\frac{2}{3}}$

7 $\sqrt{\frac{5}{7}}\ \sqrt{\frac{5}{7}}$

8 $\sqrt{\frac{1}{ab}}\ \sqrt{\frac{a}{b}}$

9 $\sqrt[3]{4}\,\sqrt[3]{16}$

10 $\sqrt[3]{5}\,\sqrt[3]{50}$

11 $(\sqrt[3]{5})^3$

12 $\sqrt[3]{a}\,\sqrt[3]{a^2}$

13 $\sqrt[3]{\frac{1}{2}}\ \sqrt[3]{\frac{1}{4}}$

14 $\sqrt[3]{a^2}\ \sqrt[3]{ab}$

15 $\sqrt[4]{x^3y}\ \sqrt[4]{xy^3}$

16 $\sqrt[4]{\frac{1}{3}}\ \sqrt[4]{\frac{1}{3}}$

17 $2\sqrt{7}\ \ 3\sqrt{21}$

18 $(3\sqrt{2})^2$

19 $(-3\sqrt{2})(-5)$

20 $3\sqrt[3]{9}\ \ \sqrt[3]{6}$

21 $-\sqrt{6}\ \ \sqrt{2a}$

22 $\sqrt[3]{\frac{a}{b}}\ \ \sqrt[3]{\frac{b^2}{a}}$

23 $\sqrt{2}\ \ \sqrt{3}\ \ \sqrt{6}$

24 $\sqrt{a}\ \ \sqrt{ab}\ \ \sqrt{b}$

Multiplication of Radicals of Different Orders [B]

Above we learned how to multiply radicals of the same order. Sometimes we need to multiply radicals of different orders. To do this the radicals must first be changed to radicals of the same order.

Example 1 $\sqrt{3} \cdot \sqrt[3]{5} = 3^{\frac{1}{2}} \cdot 5^{\frac{1}{3}} = 3^{\frac{3}{6}} \cdot 5^{\frac{2}{6}} = \sqrt[6]{3^3} \cdot \sqrt[6]{5^2}$
$= \sqrt[6]{27} \cdot \sqrt[6]{25} = \sqrt[6]{675}$

Example 2. $\sqrt[3]{3a^2}\ \sqrt[4]{4a^3} = 3^{\frac{1}{3}}a^{\frac{2}{3}}\ 4^{\frac{1}{4}}a^{\frac{3}{4}} = 3^{\frac{4}{12}}a^{\frac{8}{12}}\ 4^{\frac{3}{12}}a^{\frac{9}{12}}$

$$= \sqrt[12]{3^4\ 4^3\ a^8\ a^9}$$
$$= \sqrt[12]{3^4\ 4^3\ a^{12}\ a^5}$$
$$= a\sqrt[12]{5184\,a^5}$$

EXERCISES (8)

Multiply as indicated 1 - 16 odd

1. $\sqrt{2}\ \sqrt[3]{5}$
2. $\sqrt[3]{4}\ \sqrt{5}$
3. $\sqrt{a}\ \sqrt[3]{a^2}$
4. $2\sqrt{c}\ \sqrt[3]{c}$
5. $6\sqrt{b^3}\ 5\sqrt[3]{b}$
6. $\sqrt{a}\ \sqrt[4]{a^3}$
7. $\sqrt[3]{x^2}\ \sqrt{xy}$
8. $5\sqrt{2}\ \sqrt[3]{12}$
9. $\sqrt{3a}\ \sqrt[3]{4a^2}$
10. $\sqrt[4]{5b}\ \sqrt[3]{15b^2}$
11. $a\sqrt{b}\ b\sqrt[3]{a}$
12. $2\sqrt{2}\ \sqrt[4]{3}$
13. $\sqrt[4]{5}\ 2\sqrt{3}$
14. $\sqrt[5]{a^4}\ \sqrt[4]{a^3}$
15. $\sqrt{2xy^2}\ \sqrt[6]{4x^2y}$
16. $a\sqrt{ab}\ \sqrt[4]{2a^2b^2}$
17. $\sqrt[4]{\frac{1}{2}}\ 3\sqrt[3]{\frac{1}{4}}$
18. $5\sqrt[3]{\frac{2}{9}}\ 2\sqrt{\frac{4}{3}}$
19. $\sqrt{a-b}\ \sqrt[4]{(a-b)^3}$
20. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$
21. $(\sqrt{2}+\sqrt{5})(\sqrt{2}+\sqrt{5})$
22. $(\sqrt{7}+1)(\sqrt{7}-1)$
23. $(2-\sqrt[3]{4})(2+\sqrt[3]{4})$
24. $(3\sqrt{6}+1)(2\sqrt{6}-1)$
25. $(\sqrt{2}+\sqrt{3})^2$
26. $(\sqrt[3]{2}+\sqrt{3})^2$
27. $(\sqrt{x-2}+\sqrt{x+2})(\sqrt{x-2}-\sqrt{x+2})$
28. $(\sqrt{t+1}-3t)(\sqrt{t+1}+3t)$
29. $(\sqrt{x}+\sqrt{x-y})(\sqrt{x}-\sqrt{x-y})$
30. $(\sqrt{a+b}+\sqrt{a-b})(\sqrt{a+b}-\sqrt{a-b})$

Division of Radicals[A]

When radicals are of the same order, their quotient is the radical which is of the same order and whose radicand is the quotient of the radicands Thus

$$\frac{\sqrt{18}}{\sqrt{3}} = \sqrt{6}$$

EXERCISES [A]

In the following exercises simplify by dividing the radicand of the numerator by that of the denominator

1. $\dfrac{\sqrt{12}}{\sqrt{3}}$	5. $\dfrac{\sqrt{2}}{\sqrt{\frac{1}{2}}}$	9. $\dfrac{\sqrt[3]{36}}{\sqrt[3]{9}}$	13. $\dfrac{\sqrt{42\,y^3}}{\sqrt{\frac{1}{6}y^2}}$
2. $\dfrac{\sqrt{15}}{\sqrt{5}}$	6. $\dfrac{\sqrt{45}}{\sqrt{5}}$	10. $\dfrac{\sqrt[3]{12\,a}}{\sqrt[3]{3\,a}}$	14. $\dfrac{\sqrt[3]{72\,a^7}}{\sqrt[3]{9\,a}}$
3. $\dfrac{\sqrt{24}}{\sqrt{3}}$	7. $\dfrac{\sqrt{60}}{\sqrt{15}}$	11. $\dfrac{\sqrt[3]{2\,x^2y^3}}{\sqrt[3]{x^2y^2}}$	15. $\dfrac{\sqrt[5]{64}}{\sqrt[5]{2}}$
4. $\dfrac{\sqrt{30\,x}}{\sqrt{3\,x}}$	8. $\dfrac{\sqrt{32}}{2\sqrt{2}}$	12. $\dfrac{\sqrt[3]{9\,a^2b^7}}{\sqrt[3]{3\,ab}}$	16. $\dfrac{\sqrt[4]{a^9}}{\sqrt[4]{a}}$

Rationalizing the Denominator [A]

We say that a radical, like $\sqrt{\frac{2}{3}}$, is not in its simplest form if the radicand is a fraction We also say that a fraction, like $\dfrac{2}{\sqrt{3}}$, is not in its simplest form if it has a radical in its denominator The process of changing a fraction with a radical in its denominator to an equal expression with a rational denominator is called rationalizing the denominator

Example 1 Simplify $\dfrac{\sqrt{3}}{\sqrt{5}}$.

Solution $\dfrac{\sqrt{3}}{\sqrt{5}}=\dfrac{\sqrt{3}}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{15}}{5}$

Example 2. Simplify $\dfrac{\sqrt[3]{2}}{\sqrt{3}}$.

Solution $\dfrac{\sqrt[3]{2}}{\sqrt{3}}=\dfrac{\sqrt[3]{2}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{2^{\frac{1}{3}}\ 3^{\frac{1}{2}}}{3}=\dfrac{2^{\frac{2}{6}}\ 3^{\frac{3}{6}}}{3}=\dfrac{\sqrt[6]{2^2\cdot 3^3}}{3}=\dfrac{\sqrt[6]{108}}{3}$

Example 3 Simplify $\dfrac{1}{\sqrt{6}}$.

Solution. $\dfrac{1}{\sqrt{6}}=\dfrac{1}{\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}}=\dfrac{\sqrt{6}}{6}$

[A] **EXERCISES**

Simplify by rationalizing the denominators

1. $\frac{\sqrt{5}}{\sqrt{3}}$	4. $\frac{6}{2\sqrt{3}}$	7. $\frac{\sqrt[3]{2}}{\sqrt{3}}$	10. $\frac{\sqrt{45}}{\sqrt{8}}$	13. $\frac{2\sqrt{3}}{5\sqrt{20}}$
2. $\frac{\sqrt[3]{4}}{\sqrt{2}}$	5. $\frac{2}{\sqrt[4]{3}}$	8. $\frac{\sqrt[3]{3m}}{\sqrt{5m}}$	11. $\frac{5\sqrt{2}}{3\sqrt{75}}$	14. $\frac{2}{\sqrt[3]{4}}$
3. $\frac{3}{\sqrt{2}}$	6. $\frac{3}{\sqrt[3]{2}}$	9. $\frac{\sqrt{6}}{\sqrt[3]{9}}$	12. $\frac{\sqrt{5}}{\sqrt{14}}$	15. $\frac{5}{3\sqrt[3]{5}}$

If necessary, rationalize the denominator of each of the following fractions and then, using 1.414 as the value of $\sqrt{2}$ and 1.732 as the value of $\sqrt{3}$, find the value of each fraction

16. $\frac{\sqrt{3}}{4}$	18. $\frac{5}{\sqrt{3}}$	20. $\frac{2\sqrt{3}}{5}$	22. $\frac{2\sqrt{3}}{5\sqrt{2}}$
17. $\frac{\sqrt{2}}{3}$	19. $\frac{10}{\sqrt{2}}$	21. $\frac{\sqrt{2}}{\sqrt{3}}$	23. $\frac{5\sqrt{3}}{10}$

Rationalizing the Binomial Denominator [A]

Study the following example

Example Simplify $\frac{6}{\sqrt{2}-\sqrt{3}}$

Solution If we multiply $(\sqrt{2}-\sqrt{3})$ by $(\sqrt{2}+\sqrt{3})$, we obtain a rational number, since the product of the sum and difference of two numbers is equal to the difference of their squares. The steps in the solution are as follows

$$\frac{6}{\sqrt{2}-\sqrt{3}} = \frac{6}{\sqrt{2}-\sqrt{3}} \cdot \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \frac{6(\sqrt{2}+\sqrt{3})}{2-3}$$
$$= -6(\sqrt{2}+\sqrt{3})$$

The $(\sqrt{2}+\sqrt{3})$ is called a rationalizing factor. Two binomial surds whose product is rational, as $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$, are said to be conjugate

EXERCISES

(A)

Simplify by multiplying both terms of the fraction by the conjugate of the denominator

1. $\frac{2}{\sqrt{2}+\sqrt{3}}$ 3. $\frac{2}{\sqrt{2}+3}$ 5. $\frac{3}{\sqrt{x}-\sqrt{y}}$ 7. $\frac{\sqrt{5}-4}{2\sqrt{5}+1}$

2. $\frac{5}{\sqrt{5}-\sqrt{2}}$ 4. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ 6. $\frac{2+\sqrt{2}}{5+3\sqrt{2}}$ 8. $\frac{\sqrt{6}}{\sqrt{5}-\sqrt{3}}$

9. $\frac{2+2\sqrt{7}}{1-3\sqrt{7}}$ 10. $\frac{a}{\sqrt{a}+\sqrt{b}}$

Find the value of

11. $x^2 - 3x + 1$ when $x = \frac{3-\sqrt{5}}{2}$

12. $x^2 - 5x + 3$ when $x = \frac{5+\sqrt{13}}{2}$

13. $x^2 - 4x + 1$ when $x = 1 - \sqrt{2}$

14. $2x^2 + 3x + 5$ when $x = 1 + \sqrt{3}$

15. $3x^2 - 7x + 6$ when $x = \sqrt{3} - 2$

Introducing a Factor under the Radical Sign (A)

Sometimes computation is easier if we place the coefficient of a radical under the radical sign. To do this it is necessary to raise the coefficient to the same power as the index of the radical.

Example 1. $2\sqrt{110} = \sqrt{2^2 \cdot 110} = \sqrt{440}$

Example 2. $3\sqrt[3]{18} = \sqrt[3]{27 \times 18} = \sqrt[3]{486}$

EXERCISES

(A)

Place the coefficient under the radical sign

1. $5\sqrt{6}$ 4. $3\sqrt[3]{10}$ 7. $2\sqrt{140}$

2. $2\sqrt[3]{30}$ 5. $8\sqrt{22}$ 8. $12\sqrt{7}$

3. $5\sqrt{15}$ 6. $3\sqrt[3]{2}$ 9. $6\sqrt[3]{4}$

Complex Expressions Involving Radicals and Exponents[C]

In simplifying more difficult expressions involving radicals and exponents

1. Change from radical to fractional exponent form
2. Change negative exponents to positive exponents
3. Simplify

Example Simplify $\dfrac{\sqrt[4]{x}}{\sqrt[3]{y}}\left(\dfrac{x^{\frac{1}{4}}}{y^{\frac{1}{3}}}\right)^2 \times \dfrac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$

Solution

$$\frac{\sqrt[4]{x}}{\sqrt[3]{y}}\left(\frac{x^{\frac{1}{4}}}{y^{\frac{1}{3}}}\right)^2 \times \frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = \frac{x^{\frac{1}{4}}}{y^{\frac{1}{3}}} \times \frac{x^{\frac{1}{2}}}{y^{\frac{2}{3}}} \times \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$= \frac{x^{\frac{1}{4}}}{y^{\frac{1}{2}}} \quad \text{or} \quad \frac{\sqrt[4]{x}}{\sqrt{y}}$$

[C] **EXERCISES**

Simplify

1. $\left(\dfrac{a^{\frac{1}{2}}b^{\frac{2}{3}}}{2^{-1}c^{-2}}\right)^6$

2. $\sqrt[3]{\left(\dfrac{a^2b}{125a^{-3}b^{\frac{1}{2}}}\right)^{-1}}$

3. $\dfrac{a-b^{-1}}{ab^{-1}-b^{-2}}$ $\dfrac{a-\frac{1}{b}}{\frac{a}{b}-\frac{1}{b^2}} = b$

4. $\dfrac{a^{\frac{1}{3}}\sqrt{b^{-\frac{1}{2}}}}{b^{\frac{3}{4}}\sqrt[3]{a^{-\frac{3}{4}}}}$

5. $\sqrt{\dfrac{36\sqrt{xy^3}}{25\sqrt{x^5y}}}$ t

6. $\dfrac{\sqrt{3}+\sqrt{2}}{8^{\frac{1}{3}}-9^{\frac{1}{2}}}$

7. $\left(\dfrac{a^2b}{64a^{-3}b^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

8. $\sqrt{\dfrac{\sqrt[5]{32}\ \ \sqrt{4}}{2^{-2}-2^{-3}}}$

Irrational or Radical Equations[A]

An equation in which the variable appears under the radical sign or with a fractional exponent is called an irrational or radical equation. In solving irrational equations it is necessary to raise both members of an equation to the same power as the index of the radicals. This process may introduce numbers which are roots of the resulting equation but which are not roots of the original equation. This fact further shows the need of proving that any apparent root of an equation is an actual root.

Example 1 Solve $\sqrt{x^2+6}-x+3=0$

Solution We transform the equation so that the radical term is in one member of the equation and all other terms are in the other member

Then $\sqrt{x^2+6}=x-3$

Since the radical is of the second order, we square both members,

$$x^2+6=x^2-6x+9$$
$$6x=3$$
$$x=\tfrac{1}{2}$$

PROOF Does $\sqrt{x^2+6}-x+3=0$?
Does $\sqrt{\tfrac{1}{4}+6}-\tfrac{1}{2}+3=0$?
Does $\tfrac{5}{2}-\tfrac{1}{2}+3=0$?
Does $5=0$? *No*

Since $5\neq 0$, $\tfrac{1}{2}$ is not a root of the original equation and the equation has no solution

Example 2 Solve $\sqrt{x-5}+\sqrt{x}=5$

Solution We transform the equation so that the more complete radical term $\sqrt{x-5}$ is in one member of the equation and all other terms are in the other member

Then $\sqrt{x-5}=5-\sqrt{x}$

Squaring $x-5=25-10\sqrt{x}+x$

Transforming so that the remaining radical term is in one member and all other terms are in the other member, we have

$$10\sqrt{x}=30$$
$$\sqrt{x}=3$$

Squaring $x=9$

PROOF Does $\sqrt{x-5}+\sqrt{x}=5$?
Does $\sqrt{9-5}+\sqrt{9}=5$?
Does $2+3=5$? Yes

Therefore 9 is a root of the equation

Example 3. Solve $\sqrt[3]{x+5}=4$

Solution $\sqrt[3]{x+5}=4$

Cubing $x+5=64$

$$x=59$$

PROOF Does $\sqrt[3]{59+5}=4$? Yes

Rule for Solving Radical Equations

1 If there is but one radical term,
place it alone in one member of the equation
If there is more than one radical term,
place the more involved radical term in one member
2 Raise each member of the equation
to the same *power* as the *order of the radical*
3 If a radical remains, proceed again as in steps 1 and 2
4 Solve the resulting equation
5. Prove by substituting any apparent root
in the original equation

[A]

EXERCISES

Solve

1 $\sqrt{x} = 5$

2. $x^{\frac{1}{2}} = 4$

3 $\sqrt{x+1} = -3$

4 $\sqrt{x-5} = 4$

5. $(x-3)^{\frac{1}{2}} = 6$

6 $\sqrt[3]{x-2} = 3$

7 $\sqrt{x^2+5} + x = 5$

8 $\sqrt{y^2-11} + 1 = y$

9 $\sqrt{x} + 1 = \sqrt{x+5}$

10 $\sqrt{x} + 1 = 7 - 2\sqrt{x}$

11 $\sqrt{2y+6} = \sqrt{2y-5}$

12 $\sqrt{x} + 3 = \sqrt{x+21}$

13 $\sqrt{5x-7} = \sqrt{x+10}$

14 $\sqrt{y+7} + 4 = 0$

15 $\sqrt{x^2-8} + 4 = x$

16 $\sqrt{x+5} + \sqrt{x} = 5$

17. $2\sqrt{3x-2} = \sqrt{2x-3}$

18 $\sqrt{y} = \sqrt{7} + \sqrt{y+7}$

19 $\sqrt{x+4} + \sqrt{x-4} = 4$

20 $\sqrt{5y+4} = \sqrt{5y-9} + 1$

21 $\sqrt{x+4} + \sqrt{x-4} = 2\sqrt{x-1}$

22 $\sqrt{4x+5} - \sqrt{x+4} = \sqrt{x-1}$

23 Solve for x $\sqrt{3x-a} - \sqrt{x+a} = 0$

24 Solve for y $\sqrt{y^2+a^2+b^2-2ay} = b + a$

25 $t = \sqrt{\frac{2s}{g}}$ is a formula giving the time in seconds it takes a body starting at rest to fall a distance of s feet

a Solve the formula for s

b If $g = 32.2$, find the distance a body starting at rest will fall in 3 seconds

26. $v = \sqrt{2gs}$ is a formula giving the velocity a body will have in feet per second after falling s feet, if it starts from rest

a Solve the formula for s

b If a body starts from rest, how far has it fallen when it has attained a velocity of 44 feet per second? (Use $g = 32.2$)

27 (a) Solve $t = \pi\sqrt{\frac{l}{g}}$ for l (b) Find l when $t = \frac{1}{2}$

Imaginary Numbers [A]

Thus far we have studied real numbers, which include both rational and irrational numbers As has been stated, for each real number there is a point on the real number scale and for each point on this number scale there is a real number

You may think that there is no other kind of number Suppose you are asked to find the value of x in the equation $x^2 + 1 = 0$, in which $x^2 = -1$ The equation asks the question "What is the number whose square equals -1?" The number is not $+1$ because $(+1)^2 = +1$ The number is not -1, because $(-1)^2 = +1$ Do you see that the square of a real number cannot equal a negative number and that there is no real number which is the square root of a negative number?

To make possible the operation of expressing even roots of negative numbers, mathematicians invented numbers like $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt{-25}$, and $\sqrt{-16}$, which they called imaginary numbers

An imaginary number is an indicated square root of a negative number It follows that any even root of a negative number is imaginary

When first introduced, imaginary numbers had no practical value Now we know they are really important, especially so to the electrical engineer As has been stated, any new kind of number that is added to our number system must obey the laws of the existing number system The rules for adding, subtracting, multiplying, and dividing imaginary numbers have been so designed

The Imaginary Unit [A]

We define the imaginary unit to be $\sqrt{-1}$ We denote this unit

by italic i Then $i = \sqrt{-1}$ Any imaginary number can be written in the form $i\sqrt{a}$

NOTE In electrical engineering j is often used for $\sqrt{-1}$, because i usually denotes the amount of current

Examples $\sqrt{-6} = \sqrt{-1}\sqrt{6} = i\sqrt{6}$, $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$

In all operations with imaginary numbers we first replace $\sqrt{-}$ by i Notice the following properties of the imaginary unit

$$i^2 = \sqrt{-1}\ \sqrt{-1} = -1$$
$$i^3 = i^2\ i = -1\ i = -i$$
$$i^4 = i^2\ i^2 = -1\ -1 = +1$$

From this point on the values of i repeat in periods of 4

$$i = i^5 = i^9 = \quad \text{etc} = i$$
$$i^2 = i^6 = i^{10} = \quad \text{etc} = -1$$
$$i^3 = i^7 = i^{11} = \quad \text{etc} = -i$$
$$i^4 = i^8 = i^{12} = \quad \text{etc} = +1$$

[A]

EXERCISES

Write each of the following imaginaries as the product of i and a real number

1. $\sqrt{-4}$	5 $\sqrt{-25}$	9 $\sqrt{-100}$	13 $-\sqrt{-4}$
2 $\sqrt{-6}$	6. $\sqrt{-12}$	10 $\sqrt{-49}$	14 $-\sqrt{-5}$
3. $\sqrt{-8}$	7. $\sqrt{-\frac{1}{16}}$	11 $\sqrt{-3}$	15 $2\sqrt{-8}$
4. $\sqrt{-9}$	8 $\sqrt{-\frac{1}{25}}$	12 $\sqrt{-15}$	16 $3\sqrt{-24}$

Picturing Imaginary Numbers[A]

Let us graph any two real numbers $+A$ and $-A$ on the real number scale If we multiply $+A$ by -1, we get $-A$ (See Fig 1) We can obtain $-A$ from $+A$ by rotating OA about O through 180° We rotate the segment OA counterclockwise because a positive angle is generated by a counterclockwise rotation

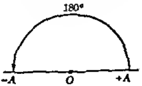


Fig 1

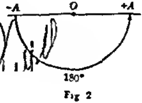


Fig 2

If we multiply $-A$ by -1, we get $+A$ (Fig 2) We can obtain the product $+A$ by rotating the line segment $O(-A)$ counterclockwise about O through 180° Thus the product of a real number and -1 can be obtained graphically by a counterclockwise rotation of 180°

Now let us try to find a graphical meaning of $\sqrt{-1}$, or i (See Fig 3) We interpret $A \cdot (-1)$ as a counterclockwise rotation of OA through 180° Since $-1 = ii$, we interpret Ai as a counterclockwise rotation of OA through 90° Then when we multiply A by i, we get Ai as shown in the diagram When we multiply the imaginary number Ai by i, we get the real number $-A$ When we multiply $-A$ by i, we rotate $O(-A)$ counterclockwise through 90° to get $-Ai$ When we multiply $-Ai$ by i, we rotate $O(Ai)$ counterclockwise through 90° to get $+A$

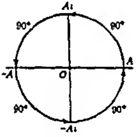


Fig 3

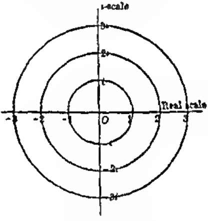


Fig 4

Since any imaginary number can be expressed as the product of a real number and i, it can be represented by a point on the line perpendicular to the real axis at O This line is called the axis of imaginaries, and the scale is called the i-scale (See Fig 4)

EXERCISES (A)

Find the values of

1. i^3	4. i^7	7. i^{12}	10. i^{10}
2. i^4	5. i^8	8. i^{16}	11. i^{14}
3. i^5	6. i^9	9. i^{20}	12. i^{18}

Draw the real and i scales having O as the origin and represent the following numbers graphically.

13. 3	16. $-4i$	19. $4i$	22. $-5i$
14. $-i$	17. $5i$	20. $-2i$	23. -5
15. -4	18. -2	21. 5	24. $6i$

Operations with Imaginary Numbers [A]

The ordinary rules of addition, subtraction, multiplication, and division apply to imaginary numbers. Do not forget to replace i^2 by -1.

Example 1

$$2\sqrt{-3}\cdot\sqrt{-5} = 2i\sqrt{3}\cdot i\sqrt{5} = 2i^2\sqrt{15} = -2\sqrt{15}$$

Example 2 $\sqrt{-6}\div\sqrt{-2} = \dfrac{i\sqrt{6}}{i\sqrt{2}} = \sqrt{3}$

Example 3 $\sqrt{12}\div\sqrt{-3} = \dfrac{2\sqrt{3}}{i\sqrt{3}} = \dfrac{2}{i} = \dfrac{2i}{i^2} = \dfrac{2i}{-1} = -2i$

Remember: The formula $\sqrt{x}\cdot\sqrt{y} = \sqrt{xy}$ does not apply when both x and y are negative. For example, $\sqrt{-2}\cdot\sqrt{-3}$ does not equal $\sqrt{6}$.

[A] **EXERCISES**

Perform the indicated operations.

1. $\sqrt{-2}\cdot\sqrt{-3}$
2. $\sqrt{-5}\cdot\sqrt{-4}$
3. $2\sqrt{-2}\cdot 3\sqrt{-5}$
4. $4\sqrt{-3}\cdot 5\sqrt{-8}$
5. $\sqrt{-2}\cdot\sqrt{-24}$
6. $\sqrt{-27}\cdot 3\sqrt{-3}$
7. $2i\sqrt{5}\cdot i\sqrt{5}$
8. $5i\sqrt{2}\cdot 3i\sqrt{18}$
9. $\sqrt{-6}\div\sqrt{2}$
10. $\sqrt{-4}\div\sqrt{-8}$
11. $\sqrt{-25}\div\sqrt{5}$
12. $\sqrt{-12}\div\sqrt{-3}$

Complex Numbers [B]

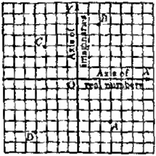
A complex number is a number having the form $a + bi$ where a and b are real numbers. The numbers $2 - 3i$, $\sqrt{5} + 2\sqrt{-3}$, and $-6 + \sqrt{-7}$ are complex. Do you see that the complex number $a + bi$ becomes a real number when $b = 0$ and an imaginary number when $a = 0$? The complex number system includes real numbers,

imaginary numbers, and the algebraic sums of a real number and an imaginary number

Since an imaginary number and a real number are unlike, their sum or difference can only be indicated

Graphical Representation of Complex Numbers[9]

Since any complex number, such as $3 - 4i$, depends on the real numbers 3 and 4, it may be represented by a point in a plane. We represent the real numbers on the horizontal axis and the imaginary numbers on the vertical axis. In the figure the complex number



$3 - 4i$ is represented by the point A. In like manner the points B, C, and D represent the complex numbers $2 + 5i$, $-3 + 3i$, and $-4 - 5i$

[8]

EXERCISES

Represent graphically the following numbers

1. $3 + 2i$	6. $3i$	11. $\sqrt{3} + i$
2. $-2 - 5i$	7. $-4i$	12. $\sqrt{5} - i\sqrt{2}$
3. $1 - 3i$	8. 5	13. $-\sqrt{6} + i\sqrt{3}$
4. $-6 + i$	9. -3	14. $\sqrt{8} + 3i$
5. $-3 - 4i$	10. $i\sqrt{2}$	15. $3 - i\sqrt{3}$

Operations with Complex Numbers[9]

A complex number is like the binomial $x + 3y$ because it has two unlike terms. We add, subtract, and multiply complex numbers as we do binomials. Study Example 1 for addition and subtraction of complex numbers, Example 2 for multiplication of them, and Example 3 for division of them

Example 1 Add $4+3i$ and $-6-i$

Solution We combine the real numbers getting -2 and combine the imaginary numbers getting $+2i$

$$\begin{array}{l} 4+3i \\ \underline{-6-i} \\ -2+2i \quad \text{the sum} \end{array}$$

Example 2 Multiply $3-4i$ by $2+3i$

Solution We multiply each term of $3-4i$ by each term of $2+3i$ and combine like terms getting $6+i-12i^2$ Since $i^2=-1$ $-12i^2=12$ Then $6+i-12i^2$ $=6+i+12=18+i$ the product

$$\begin{array}{l} 3-4i \\ \underline{2+3i} \\ 6-8i \\ \underline{\quad +9i-12i^2} \\ 6+\ \ i-12i \quad \text{the product} \end{array}$$

Example 3 Divide $4-2i$ by $1+3i$

Solution We first write the quotient as a fraction getting $\frac{4-2i}{1+3i}$ Then in a manner similar to that used in rationalizing a binomial denominator of a fraction we multiply both numerator and denominator of the fraction by $1-3i$ as follows

$$\frac{4-2i}{1+3i}\times\frac{1-3i}{1-3i}=\frac{-2-14i}{1^2-(3i)^2}=\frac{-2-14i}{10}$$

$$=\frac{-1-7i}{5} \quad \text{the quotient}$$

The binomials $1+3i$ and $1-3i$ are called <u>conjugate complex numbers</u>

[B] **EXERCISES**

Combine as indicated

1 $(3-2i)+(1-i)$
2 $(7+4i)+(-3+5i)$
3 $(-4-i)+(-5+i)$
4 $3+(7-2i)$
5 $-8-(1-i)$
6 $(7+3i)-(2+2i)$
7 $(8+5i)-(5+i)$
8 $(6-2i)+(3-4i)$
9 $(-2-i)-(7+4i)$
10 $(-3-4i)+(i-5)$

Multiply as indicated and simplify each product

11 $(3+2i)(1-4i)$
12 $(5-i)(5+i)$
13 $+10(4-3i)$
14 $-8(2-i)$
15 $(4-i)(4+i)$
16 $(-3+5i)(-3-5i)$
17 $(a+bi)(a-bi)$
18 $(x+iy)(x-iy)$

Divide as indicated and simplify each quotient

19 $(3-4i) \div (3+4i)$ 23 $(a+bi) \div (a-bi)$

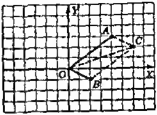
20 $(1-i) \div (1+i)$ 24 $(x-yi) \div (x+yi)$

21 $8 \div (1+2i)$ 25 $(5-4i) \div (5+4i)$

22 $-4 \div (2+i)$ 26 $(10-i) \div (10+i)$

Graphical Addition and Subtraction of Complex Numbers [8]

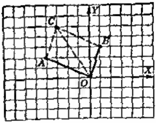
To add graphically two complex numbers, such as $4+3i$ and $2-i$, represent the numbers by the points A and B respectively, and draw OA and OB as shown at the right. Then complete the parallelogram having adjacent sides OA and OB. The parallelogram is $OACB$, and point C represents the sum of the two complex numbers. Graphically the sum is $6+2i$. We can check by adding the numbers algebraically.



To subtract two complex numbers, we change the sign of the subtrahend and proceed as in addition. The process is illustrated by the following example.

Example Subtract $4-2i$ from $1+3i$.

Solution Changing the subtrahend to $-4+2i$, we have to add $-4+2i$ and $1+3i$. We represent the numbers by points A and B respectively, as shown in the diagram. Draw OA and OB. Then $OACB$ is the completed parallelogram and C represents the difference of the complex numbers. Graphically the difference is $-3+5i$. We can check by subtracting the numbers algebraically.



EXERCISES [8]

Perform the indicated additions and subtractions graphically

1 $(1+2i)+(3+i)$ 3 $(-3+i)+(-1-2i)$

2 $(2+i)+(-3+2i)$ 4 $(2-i)+4i$

5. $(3 + i) + (-4 + 2i)$	9. $(3 - 2i) - (2 - i)$
6. $5 + (-4 + 3i)$	10. $3 - (1 - 3i)$
7. $(-2 + 3i) - (3 + 2i)$	11. $4i - (i - 2)$
8. $(5 + i) - (1 + 3i)$	12. $(2 - 3i) - 3$

Checking Your Understanding of Chapter 8

You have completed Chapter 8, except the reviews and tests. Before you leave this chapter, be sure that you know

Your chance to review

	PAGE
1. The laws of exponents	203
2. The meaning of a fractional exponent	208
3. How to evaluate radicals	206
4. The value of the zero power of any number, except 0	209
5. How to simplify radicals	216–218
6. How to combine radicals	218–219
7. How to multiply and divide with radicals	219–222
8. How to plot imaginary numbers	229
9. How to spell and use correctly the following words	

MATHEMATICAL VOCABULARY

	PAGE		PAGE
complex number	231	radical	204
conjugate	223	radical equation	225
imaginary number	228	rational number	213
index	205	standard notation	211
irrational number	214	surd	214
principal root	205		

If you are trying to do better than average work, be sure that you know

	PAGE
10. How to plot complex numbers	232
11. How to add, subtract, multiply, and divide with complex numbers	232
12. How to add and subtract complex numbers graphically	234

CHAPTER REVIEW

Perform the indicated operations and simplify: (A)

1. $x^3 \cdot x^{n-3}$
2. $5^{x+1} + 5^{x-1}$
3. $(-3)^2(-3)^3$
4. $2(2x^n)^2$
5. $(a^{-\frac{1}{3}})^3$
6. $a^{\frac{3}{4}} \cdot a^{\frac{1}{4}}$
7. $x^0 \cdot x^{n-2}$
8. $m^{\frac{1}{2}} + m^{\frac{1}{3}}$
9. $\left(\frac{a^{2n}}{b^n}\right)^3$
10. $\left(\frac{a^x}{a^{x-1}}\right)^x$
11. $\left(\frac{x^{m-2}}{x^m}\right)^3$
12. $\left(\frac{c^{2n+1}}{c^{n-1}}\right)^2$

Write with positive exponents

13. ab^{-2}
14. $2a^{-1}b^x$
15. $3a^nb^{-n}$

Evaluate

16. $(-27)^{\frac{1}{3}}$
17. $(\frac{64}{27})^{-\frac{2}{3}}$
18. $(\frac{9}{25})^{-\frac{1}{2}}$

Express without radicals

19. $\sqrt[3]{a^2}$
20. $4\sqrt{a^3}$
21. $\sqrt[4]{b}$
22. $\sqrt[3]{m} \cdot \sqrt{4m^3}$
23. $\sqrt{(a+b)^3}$
24. $\sqrt[3]{m^{\frac{3}{2}}}$

Write in scientific notation

25. 86,000,000
26. 0.060
27. 170,000,000

Simplify

28. $\sqrt{27}$
29. $\sqrt[3]{54}$
30. $\sqrt{\frac{9a}{8}}$
31. $\sqrt[4]{36}$
32. $\sqrt{\frac{1}{2}}$
33. $\sqrt{\frac{25}{128}}$
34. $\sqrt[3]{256}$
35. $\sqrt[3]{\frac{9}{16}}$
36. $\sqrt{\frac{16m^3}{a^5}}$

Add

37. $\sqrt{24} - 3\sqrt{54} + \sqrt{96}$
38. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} - \sqrt{\frac{9}{8}}$

Rationalize the denominators

39. $\frac{2}{\sqrt{3}}$
40. $\frac{5}{3\sqrt{2}}$
41. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{5}+\sqrt{3}}$

Multiply and divide as indicated

42. $\sqrt{3} \cdot \sqrt{2}$
43. $\sqrt[3]{8} \div \sqrt{2}$
44. $\sqrt{-3}\sqrt{-5}$
45. $\sqrt[3]{4} \cdot \sqrt{6}$
46. $(2\sqrt[3]{12})(3\sqrt[3]{2})$
47. $\sqrt[4]{28}\sqrt[4]{20}$

(B)

Multiply as indicated

8 $\sqrt{(a-b)}\ \sqrt[3]{(a-b)^3}$ 50 $4\sqrt[3]{\frac{1}{9}}\ 2\sqrt{\frac{4}{3}}$

9 $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ 51 $\sqrt{3ab^2}\ \sqrt[6]{9a^3b}$

Solve

52. $\sqrt{x+1}=1+\sqrt{x-2}$

53 $\sqrt{x+9}+\sqrt{x-3}=2\sqrt{x+2}$

Do as indicated

4 $i(4-2i)$ 56 $(2-3i)\ (4+3i)$

5 $-6(7-5i)$ 57. $(5-2i)\div(5+2i)$

58 Add $3+2i$ and $4-3i$ graphically

(A) CUMULATIVE REVIEW

1. Simplify $2a-(3a-b)-(b-2a)$
2. Subtract $3a+2b-3c$ from $4a-2b-5c$
3 Multiply $(a^2-b^2)(a^2-2ab+b^2)$
4 Divide $81-3y^4+9y^2+12y^3$ by $3y-y^2+9$

Write the special products

5. $(2a-b)^2$ 8 $(a+b-c)^2$

6 $(3x-2y)(4x+y)$ 9 $(3x-y-2z)^2$

7. $(5a-b)(5a+b)$ 10. $(a+b-c)(a+b+c)$

Factor

11. $6ax-3a^2y^2$ 16 $1-(x-2)^2$

12. x^2-9y^2 17 $2a^3-4a^2+a-2$

13 x^3-8y^3 18. x^4-13x^2+36

14. $8a^3+27b^3$ 19. $x^2-ax-ab+bx$

15. $16x^4-81y^4$ 20 a^2-b^2+a-b

Reduce

21 $\dfrac{a^2-4b^2}{a^3-8b^3}$ 22 $\dfrac{b^3+8}{b^3+2b^2-4b-8}$

Combine

23. $a+b-\dfrac{a^3-b^3}{a^2-ab+b^2}$ 24. $\dfrac{3-x}{x^2+2x+1}+\dfrac{2+x}{x^2+x}$

Simplify

25 $\dfrac{1+\dfrac{2}{x-1}}{\dfrac{x^2+x}{x^2+x-2}}$ 26 $\dfrac{\dfrac{1}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}$

Solve

$$27\quad \frac{6}{x+2}-\frac{x+2}{x-2}-\frac{x^2}{4-x^2}=0$$

28 $x+2y+4=0$
$4y-2x-8=0$

29 $\frac{1}{2}x-\frac{1}{3}y=1$
$y=3x$

30 Five tons of ore that is 25% zinc are obtained by mixing ore that is 30% zinc with an ore that is 17.5% zinc. How many tons of each ore must be used?

31 The length of a rectangle exceeds its width by 6 inches. If the perimeter is 72 inches, find the length and width.

32 The sum of the digits of a number of two figures is 14. If the digits are reversed, the number is increased by 36. Find the number.

CUMULATIVE TEST [illegible]

Solve each problem and write the answer (*a*), (*b*), (*c*), or (*d*) which you think is correct. Give only one answer for each problem.

1 Remove parentheses and simplify

$$4a-3(a-1)-\{a-(1+a)\}$$

a $a+2$ c $a+4$
b $a-2$ d $3a-2$

2 Find the value of $\dfrac{3a^2}{b}-b^3$ when $a=3$, $b=-2$

a $-21\frac{1}{2}$ c $-5\frac{1}{2}$
b $21\frac{1}{2}$ d $-356\frac{1}{2}$

3 Find the root of $\dfrac{2x+1}{3}-\dfrac{x+4}{4}=1$

a [illegible] c $2\frac{2}{5}$
b [illegible] d 4

4. The time for a plane to fly a certain distance with the wind is 3 hr and the time returning against the wind is 3 hr. 30 min Find the velocity of the wind if the speed of the plane in still air is 350 m p h

a. 10 m p h c about 27 m p h
b. 40 m p h d about $11\frac{1}{2}$ m p h

5. Multiply $3x - y + 1$ by $3x + y - 1$

a. $9x^2 - y^2 - 1$ c $9x^2 + y^2 + 6xy - 1$
b. $9x^2 - y^2 + 2y - 1$ d $9x^2 - y^2 - 1$

6. Factor $x^4 + x^2 + 1$

a. $(x^2 + x + 1)(x^2 - x - 1)$ c $(x^2 + x + 1)(x^2 - x + 1)$
b. $(x^2 - x + 1)(x^2 - x - 1)$ d $(x^2 + x - 1)(x^2 + x + 1)$

7. Find the prime factors of $a^6 - b^6$

a $(a^2 - b^2)^3$
b $(a + b)(a - b)(a^2 - 2ab + b^2)$
c. $(a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$
d $(a + b)(a - b)(a^2 - 2ab + b^2)(a^2 + 2ab + b^2)$

8. Find the root of $\dfrac{x-5}{x-3} + \dfrac{x-2}{x+2} = 2$

a. 3 c. $-1\frac{1}{3}$
b -2 d $1\frac{1}{3}$

9. Simplify $\dfrac{a^2 - \dfrac{1}{a}}{a^3 - \dfrac{1}{a^3}}$

a $\dfrac{a^2}{a^3 + 1}$ c $\dfrac{1}{a^2}$
b. $\dfrac{a^3 + 1}{a^2}$ d $\dfrac{a^2 + 1}{a^2}$

10 Solve $\dfrac{x}{x-3} + \dfrac{3}{x+3} = \dfrac{x^2+9}{x^2-9}$.

a. The root is 3 c. The root is 0.
b The root is -3 d There is no root

11. A tank can be filled by one pipe in x hours and by another pipe in y hours It can be filled by both pipes in

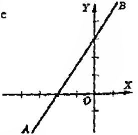
a $\frac{x+y}{xy}$ hours

b $\frac{xy}{x+y}$ hours

c. $\frac{1}{x+y}$ hours

d. $x+y$ hours

12. What is the equation of the line AB in the figure at the right?



a. $y = \frac{2}{3}x - 2$

b $y = \frac{3}{2}x + 3$

c. $y = \frac{2}{3}x + 2$

d. $y = \frac{-3x}{2} - 2$

13. $f(x) = b^2 - x^2$ Find $f(1-b)$

a. $2b - 1$

b. $1 - 2b$

c. $2b^2 + 2b - 1$

d. $2b^2$

14. $F = \frac{Wv^2}{gr}$. How is F changed when W is unchanged, v is doubled, g is unchanged, and r is trebled?

a. Multiplied by $\frac{4}{3}$

b. Divided by $\frac{4}{3}$

c. Multiplied by $\frac{2}{3}$

d. Divided by $\frac{2}{3}$

15. Form the linear equation which expresses the relation between x and y in the table

x	0	1	3	5
y	−4	−1	5	11

a. $y = 4 - 3x$

b. $y = 3x + 2$

c. $x + y = 8$

d $3x - 4 = y$

16. The graphs of $2y = -3x - 8$ and $3x + 2y = 2$

a. have the same x-intercept

b. have the same y-intercept

c. intersect each other

d. have equal slopes

17 A man receives \$276 annually on investments of \$4500 and \$3200 The rate of interest on the \$4500 investment is

1% greater than the rate on the \$3200 investment What is the rate on the \$4500 investment?

a 2% c 3%
b. $3\frac{1}{2}$% d 4%

18 Solve the set of equations

$$x + y + z = 2$$
$$2x - y - z = 1$$
$$x + 2y + 3z = 5$$

a. $x = 1, y = 1, z = 0$ c $x = 1, y = -1, z = 2$
b. $x = 1, y = -1, z = 0$ d $x = 0, y = 1, z = 1$

19. P varies inversely as x^2 and directly as y If $P = 9$ when $x = 2$ and $y = 12$, find the value of P when $x = 5$ and $y = 6$

a. $2\frac{1}{12}$ c $\frac{18}{25}$
b $3\frac{3}{4}$ d $\frac{12}{25}$

20 If 10 men working 8 hours a day can complete a piece of work in 18 days, how many days will be needed for 12 men working 9 hours a day to do the work?

a 15 days c 6 days
b 16 days d $13\frac{1}{3}$ days

21 Rationalize the denominator of $\dfrac{\sqrt{5} - 1}{2\sqrt{5} + 1}$.

a. $\dfrac{11 - 3\sqrt{5}}{9}$ c. $\dfrac{11 - 3\sqrt{5}}{19}$

b $\dfrac{51 - 3\sqrt{5}}{49}$ d. Not (a), (b), or (c)

22 Simplify $\sqrt[m]{\dfrac{a^{m+2}}{b^{m-2}}}$.

a. $\dfrac{a}{b}\sqrt[m]{a^2b^2}$ c $ab\sqrt[m]{\dfrac{a^m}{b^m}}$

b $a^2b^2\sqrt{\dfrac{a^m}{b^m}}$ d $\dfrac{1}{b}\sqrt[m]{a^{m+2}b^2}$

23. Simplify $\frac{x^{1-a}}{x^{a-1}}$.

a. x^2

c. $x^a x^2$

b. $\frac{x^2}{x^{2a}}$

d. $\frac{x^a}{x^2}$

24. The base of a rectangle is $2\sqrt{2} + 2$ and the area is 12 Find the perimeter

a. $8\sqrt{2}$

c $12(2\sqrt{2} - 1)$

b. $24(\sqrt{2} - 1)$

d. Not (a), (b), or (c)

25. Multiply $1 - 3i$ by $1 + 2i$

a $-5 - i$

c. 7

b. $7 - i$

d. $1 - 6i$

CHAPTER TEST

[A]

Perform the indicated operations

1 $(-2)^3$

2 $(2x^2)^n$

3. $\left(\frac{a^2}{b}\right)^3$

4. $\frac{c^{2n}}{c^n}$

5 $a^2 \cdot a^n$

6. $a^{\frac{1}{3}} \cdot a^{\frac{2}{3}}$

7. $\left(\frac{x^n}{y}\right)^2$

8 $\frac{x}{x^{\frac{1}{3}}}$

9 $y^{\frac{1}{2}} \cdot y^{\frac{3}{2}}$

10 $100^{\frac{1}{2}} \cdot \sqrt{25}$

11 $\sqrt{\sqrt[3]{64}}$

12 $m^{\frac{1}{2}} \div m^{\frac{1}{4}}$

Evaluate

13 $4^{\frac{1}{2}}$

14. $125^{\frac{2}{3}}$

15. $(25)^0 + \sqrt{25}$

16. Write 0 050 in scientific notation

17. Write without an exponent $2\ 6 \times 10^5$

Reduce to lower order

18. $\sqrt[6]{81}$

19. $\sqrt[4]{9x^2}$

20 $\sqrt[6]{8x^3}$

Multiply

21. $\sqrt{3}\sqrt{5}$

22 $5\sqrt{2} \cdot \sqrt{6}$

23. $\sqrt[3]{4x} \cdot \sqrt[3]{2x^2}$

Divide

24 $\sqrt{24} \div \sqrt{2}$

25. $\sqrt{60} \div \sqrt{15}$

26. $\sqrt[3]{24a^6} \div \sqrt{3a}$

Simplify

27. $\sqrt{45}$ 28. $\sqrt{\frac{2}{3}}$ 29 $\sqrt{48x^2y}$ 30 $\sqrt[3]{81a^4}$

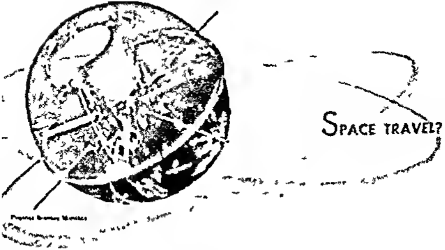
31 Is 6 a root of $\sqrt{x-2}=x-8$?

32. Solve $\sqrt{3x+4}=\sqrt{3x-11}+1$

Do as indicated

33. $\sqrt{-6}\cdot\sqrt{-2}$ 34. $\sqrt{-8}\div\sqrt{2}$ 35 $i^2\ \ i^3$

36 Simplify $\dfrac{2}{\sqrt{2}-\sqrt{3}}$



SPACE TRAVEL?

If you are observing the heavens some evening shortly after sunset or in the morning just before sunrise you may be able to see a faint object near the horizon. This object will not be a flying saucer but a man-made satellite—a man-made moon.

Many such satellites have been planned by the United States Committee for the International Geophysical Year (1957–1958). The committee plans to use the satellites for a study of outer space. It plans to use the satellites, which are equipped with very delicate instruments, in making several major experiments. By these experiments it hopes to learn more about the density of the outer atmosphere the air drag, the cosmic rays, the composition of the earth's crust, the earth's shape and the micro-meteorites.

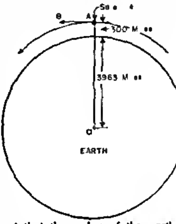
What are these satellites and why will they fly around in space? Let us take a look at one of them. It is a sphere about 20 inches in diameter and weighs about 20 pounds. It is filled with very small and delicate scientific instruments which can signal data back to the earth. It has an approximate speed of 18 000 miles an hour encircling the earth in an elliptical orbit (path) 16 times daily (Model pictured above)

A three stage rocket is to be used to shoot the satellite into its orbit and give it the needed velocity to stay aloft. The first stage of the rocket is a modification of the Martin Viking. It has a greater thrust than the combined pull of hundreds of heavy draft horses. The second stage of the rocket is used to boost the velocity. The third stage besides adding to the vertical velocity, delivers a side thrust, ejecting the satellite at an approximate speed of 18 000 miles an hour

Now let us find what velocity is needed to keep the satellite aloft in its orbit. For it not to return to the earth, it must have a velocity whose component parallel to the earth's surface will overcome gravity

In other words it must have a centrifugal force equal to the force exerted on it by the earth.

Let us assume that the satellite is 300 miles above the earth's surface and that it is in space free from atmosphere and other forms of matter. With these assumptions let us find the velocity of the satellite at A in the direction AB which is needed to overcome gravity. At 300 miles up the value of gravity is less than at the earth's surface.



Let d_1 and d_2 denote two distances from the earth's center and let g_1 and g_2 denote the two corresponding values of gravity. By Newton's universal law of gravitation $\frac{g_1}{g_2} = \frac{d_2^2}{d_1^2}$. Suppose the value of gravity at a point on the earth's surface is 32.09 and that the radius of the earth is 3963 miles.

$$\text{By the above formula, } \frac{g_1}{32.09} = \frac{3963^2}{4263^2}$$

$$\text{Solving } g_1 = 27.73 \text{ (ft./sec}^2\text{)}$$

From a study of mechanics $g_1 = \frac{v^2}{r}$ and $v = \sqrt{g_1 r}$. Here v is the velocity in feet per second in the direction AB, r the distance in feet from the center of the earth, and g_1 the value of gravity at A.

$r = 4263$ mi. or 22,508,640 ft.

Then $v = \sqrt{27.73 \times 22{,}509{,}000}$
$= 24{,}983$ feet per second
or 17,034 miles per hour

Then if a satellite 300 miles up has a velocity of about 17,000 miles an hour in the direction AB, its centrifugal force equals its attraction to the earth and it has no weight. This is the condition for it to stay aloft in free space.

By solving the differential equation $x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0$ it can be shown that the satellite's orbit is an ellipse having one of its two foci at the center of the earth.

These small satellites may prepare the way for space ships which man may use to reach the moon. Who knows?

CHAPTER 9

Quadratic Functions
nd Equations

In this chapter you will study
uations not of the first degree

As you know, we studied linear functions and equations in Chapter 5 These functions and equations contained only the first powers of the unknown and their graphs were straight lines

Now we shall study quadratic functions and equations which have many properties not common to linear equations

A quadratic function is one of the second degree x^2 $2x^2 - 6x$ and $-3x^2 + 7x - 1$ are quadratic functions of x Any function of the form $ax^2 + bx + c$ in which a b and c are constants and a is not zero is a quadratic function of x

If we let some letter as y equal a quadratic function we obtain a quadratic equation in x For example $y = x^2 - 3x$ is a quadratic equation in x

Let us make a study of the quadratic function $2x^2 - 5x + 4$ and see how the function changes when x increases from -3 to $+6$ Below is a table containing sets of values of x and the function of x

If $x =$	-3	-2	-1	0	1	2	3	4	5	6
then $2x^2 - 5x + 4 =$	37	22	11	4	1	2	7	16	29	46

55
50
67/
6
20
36
16
2

[A] **EXERCISES**

1 In the table above each value of x is greater by 1 than the preceding value What can you say of the successive values of the function?

2 Does an increase in x always produce an increase in the function? Apply this question to a linear function

3 For what values of x in the table above does the function increase when x increases? For what values of x does the function decrease when x increases?

4 Complete If any increase in x produces a proportional increase in a function of x the graph of the function is a ?
?

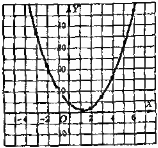
How to Graph a Quadratic Function [A]

The graph of a function usually gives a better understanding of the function than does a table of values Let us graph the function $2x^2 - 5x + 4$ which we have just studied We can let y or $f(x)$ represent the function We shall let $y = 2x^2 - 5x + 4$ We first make a table of values

In making the table we assign values to x and compute the values of y. Note that we assign negative values as well as positive values to x.

$$y = 2x^2 - 5x + 4$$

x	-3	-2	-1	0	1	2	3	4	5	6
y	37	22	11	4	1	2	7	16	29	46



When we have completed the table, we plot the points whose coördinates are in the table. Finally we draw a smooth curve through the points, taken in order.

Notice that no part of the graph is a straight line. For this reason many points must be plotted before the graph can be drawn. As x increases from -3 to $1\frac{1}{4}$, the function decreases, and as x increases from $1\frac{1}{4}$ to 6, the function increases.

This curve is called a parabola. An axial section of an automobile headlight is a modified parabola. The suspension cables of some of our largest suspension bridges are parabolas.

EXERCISES (A)

Draw the graphs of the following functions.

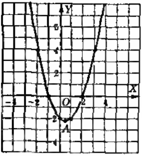
1. $x^2 + x - 6$
2. $x^2 - x - 12$
3. $2x^2$
4. $-6x^2$
5. $-x^2$
6. $8 - x - x^2$
7. $x^2 - 9$
8. $2x^2 + 3x - 5$
9. $2x^2 - 7x + 4$

Terms Associated with the Graph[A]

Shown at the top of the opposite page is the graph of $x^2 - x - 2$, or of the equation $y = x^2 - x - 2$.

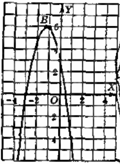
Note that the graph intersects the y-axis in the point $(0, -2)$. Then the y-intercept is -2.

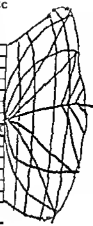
For what value of x does $y = -2$?



The point A on the graph is called the turning point of the curve. At A the function changes from a decreasing function to an increasing function (see page 133). At the turning point, $x = \frac{1}{2}$ and $y = -2\frac{1}{4}$. Do you see that the smallest value of y is $-2\frac{1}{4}$? The smallest value of a function is called the minimum value. Then the minimum value of $x^2 - x - 2$ is $-2\frac{1}{4}$. This curve is concave upward, or U-shaped.

Remember. If a curve is U-shaped, it is concave upward. Note that the coefficient of x^2 in the function $x^2 - x - 2$ is positive.





Now let us study the graph of the function $-2x^2 - 5x + 3$, or of the equation

$$y = -2x^2 - 5x + 3$$

The y-intercept is $+3$. How can you find the y-intercept without seeing the graph? The turning point of the graph is at $B(-1\frac{1}{4}, +6\frac{1}{8})$. At the point B, the function changes from an increasing function to a decreasing function. The curve is concave downward.

The y-Intercept [A]

The general form of a quadratic function of x is $ax^2 + bx + c$, the coefficient a not being zero.

The graph of $y = ax^2 + bx + c$ crosses the y-axis at the point whose x value is zero. Then the y-intercept can be found by substituting zero for x in the equation $y = ax^2 + bx + c$.

Let $x = 0$. Then $y = 0 + 0 + c$. Then the y-intercept of the graph of $y = ax^2 + bx + c$ is c, the constant term.

[A] **ORAL EXERCISES**

Find the y-intercepts of the graphs of

1. $y = 2x^2 - 3x - 1$
2. $y = -3x^2 + 2x - 3$
3. $y = x^2 + x$
4. $y = 3x^2$
5. $y = 7 - x - x^2$
6. $y = 5 - x^2$
7. $2x^2 - 8x - 4 = 0$
8. $1 - 3x + x^2 = 0$
9. $-2 + 7x^2 = 0$

The Maximum and Minimum Values [A]

For each value of x in the equation $y = ax^2 + bx + c$ there is one value of y. We shall now show that y has a maximum or minimum value when x has the value $-\frac{b}{2a}$

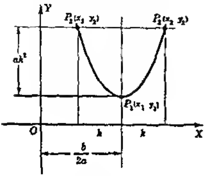


Fig 1

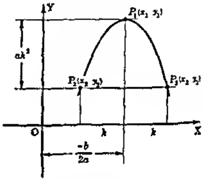


Fig 2

Let x_1 and y_1 denote a pair of real values of $y = ax^2 + bx + c$ and let P_1 denote the point (x_1, y_1) on the graph (Figs 1 and 2)

Let $$x_1 = -\frac{b}{2a} \tag{1}$$

Then $$y_1 = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$

Simplifying $$y_1 = \frac{-b^2}{4a} + c \tag{2}$$

Let x_2 and y_2 denote another pair of values of the equation and let P_2 denote the point $(x_2\ y_2)$ on the graphs

Let $k = x_2 - x_1$ Then $x_2 = x_1 + k$

From (1), $x_2 = \frac{-b}{2a} + k$

Then $$y_2 = a\left(\frac{-b}{2a} + k\right)^2 + b\left(\frac{-b}{2a} + k\right) + c$$

Simplifying, $$y_2 = \frac{-b^2}{4a} + ak^2 + c \tag{3}$$

Subtracting (2) from (3) $$y_2 - y_1 = ak^2 \tag{4}$$

Now let us see what equation (4) tells us When k is either positive or negative, k^2 is positive

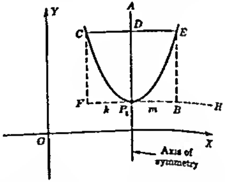
1 When a is positive Since k^2 is positive, ak^2 is positive when a is positive Then when x_1 is either increased or decreased by any real number k, the value of y is increased by ak^2 This means that y has a minimum value when $x = \frac{-b}{2a}$ that the graph is concave upward, and that there is a turning point at P_1

2 When a is negative When the coefficient a of k^2 is negative, ak^2 is negative Then when x_1 is either increased or decreased by the real number k, the value of y is decreased by ak^2 This means that y has a maximum value when $x = \frac{-b}{2a}$ We also know that the curve is concave downward and has a turning point at P_1

The maximum or the minimum value of y in the equation $y = ax^2 + bx + c$ can be found by substituting $\frac{-b}{2a}$ for x in the equation or by formula (2) which is $y = \frac{-b^2}{4a} + c$

The Axis of Symmetry [A]

The axis of symmetry of a curve is the line which bisects any line segment perpendicular to it and having its end points on the curve



In the figure shown here P_1 is the turning point of the parabola which is the graph of $y = ax^2 + bx + c$ (a being positive)

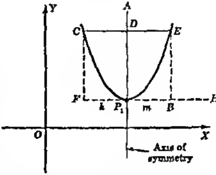
Through P_1 draw $P_1A \parallel OY$ and $P_1H \parallel OX$

Let CDE be any line segment $\perp P_1A$ intersecting the curve in points C and E

Draw

$CF \perp FP_1$

and $EB \perp P_1B$



Now $CF \parallel FB$ and $CF \parallel EB$

Let $FP_1 = k$ and $P_1B = m$

When the value of x at P_1 is decreased by k, the value of y is increased by ak^2, and when it is increased by m, the value of y is increased by am^2

Since $CF = BE$, $ak^2 = am^2$

Then $k = m$ and $FP_1 = BP_1$

Then $CD = DE$

Since P_1A bisects any line segment $\perp$ to it and having its end points on the curve, it is the axis of symmetry of the curve

Since the value of x at P_1 is $\frac{-b}{2a}$ and $P_1A \parallel OY$, the equation of the axis of symmetry is

$$x = \frac{-b}{2a} \quad \text{or} \quad 2ax + b = 0$$

The same reasoning applies when the curve is concave downward

Summary Given $y = ax^2 + bx + c$

1 The value of x at the turning point of the graph $= \frac{-b}{2a}$

2 If a is positive, the minimum value of the function can be found by substituting $-\frac{b}{2a}$ for x in the equation or by finding the value of $-\frac{b^2}{4a} + c$.

3 If a is negative, the maximum value of the function can be found by substituting $-\frac{b}{2a}$ for x in the equation or by finding the value of $-\frac{b^2}{4a} + c$

4 The equation of the axis of symmetry is $x = -\frac{b}{2a}$ or $2ax + b = 0$

Example 1 *Find the maximum or minimum value of the function* $-x^2+2x-4$

Solution The coefficient of x^2 in the function is negative Then the graph of the function is concave downward, and the function has a *maximum value* *The value of* x *at the turning point* $= -\frac{b}{2a} = -\frac{2}{-2} = 1$

The value of the function at the turning point $= f(1) = -1+2-4 = -3$, or $\frac{-b^2}{4a}+c = 1-4 = -3$

So the maximum value of the function is -3

Example 2 Find the equation of the axis of symmetry of the graph of $y = 3x^2 - x + 1$

Solution $a = 3$ and $b = -1$ Substituting these values in $2ax + b = 0$, we get $6x - 1 = 0$, the equation of the axis of symmetry

EXERCISES

[A]

Find the turning points of the graphs of the following functions

1. x^2+x-6	3 x^2	5 $12-x-x^2$
2 $6+x-x^2$	4 $-3x^2$	6 $2-5x-3x^2$

Find the maximum or minimum values of the following functions

7. x^2-4x	9 $-2x^2+6x$	11 $4-3x-x^2$
8. x^2+2x	10 x^2+4	12 $6x^2-8x+1$

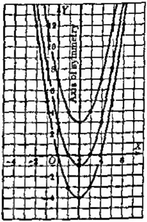
Find the equations of the axes of symmetry of the graphs of

13 $y=x^2-x+1$	15. $y=-2x^2+3$	17. $y=4-5x^2$
14 $y=2x^2-5x-1$	16 $y=x^2$	18 $y=1-x^2$

How the Constants of the Function Affect the Graph [A]

As you have learned, the sign of the coefficient of x^2 in the function ax^2+bx+c tells whether the curve is concave upward or concave downward The coefficients a and b determine the shape of the curve In general, the branches of the curve become closer to each other when a increases and b remains constant

For example, the graph of $4x^2 + 2x - 6$ has its branches closer together than the graph of $2x^2 + 4x - 6$. The constants a and b, then, determine the shape of the graph.



Let us see how the constant c affects the graph. Here are the graphs of the three functions $x^2 - 4x$, $x^2 - 4x + 3$, and $x^2 - 4x + 7$. All three functions have the same values of a and b and have different values of c. The turning points of the graphs are $(2, -4)$, $(2, -1)$, and $(2, 3)$. The line $x = 2$ is the axis of symmetry of each curve.

If the lowest curve is raised 3 units, it will coincide with the middle curve, and if it is raised 7 units, it will coincide with the highest curve. Since all three curves are congruent, we may infer that the constant c does not affect the shape of the graph but does affect the position of the graph with reference to the x-axis.

Quadratic Equations in One Variable (A)

The degree of a rational integral equation, when reduced to its simplest form, is the degree of the term or terms of the highest degree. Thus the equation $4x^3 + xy^4 + 7xy + 6 = 0$ is of the third degree in x, the fourth degree in y, and the fifth degree in x and y. An equation of the first degree is a linear equation; one of the second degree is a quadratic equation; and one of the third degree is a cubic equation.

A complete quadratic equation in one variable is one that contains both the first and second powers of the variable, as $2x^2 - 5x + 8 = 0$ and $x^2 + 3x = 2$.

An incomplete, or pure, quadratic equation is one that contains only the second power of the variable, as $6x^2 - 216 = 0$ and $5x^2 = 125$.

As you have learned, the general form of a quadratic equation is $ax^2 + bx + c = 0$, in which a, b, and c are constants and a is not zero. If a in the equation $ax^2 + bx + c = 0$ equals zero, the equation becomes the linear equation $bx + c = 0$.

There are four methods of solving quadratic equations, which are as follows:

1 Solution by factoring, when the roots are rational
2 Solution by graphs, when the roots are real
3 Solution by finding the square roots
4 Solution by formula

We studied the solution by factoring in Chapter 3

Roots of a Quadratic Equation [A]

Any quadratic equation has two, and only two roots These two roots are both real or both imaginary You are not prepared to prove these facts at this time, but you will find them useful to know

How to Solve Quadratic Equations by Graphs

A very useful application of graphs of functions is in the solution of equations which contain these functions Graphs also enable one to obtain a better understanding of equations

Suppose that we wish to solve the quadratic equation $2x^2 - x = 28$ by graph If we write this equation in the form $2x^2 - x - 28 = 0$, the left member remains a quadratic function of x, and its graph is a parabola Solving $2x^2 - x - 28 = 0$ means finding the two values of x that make the left member equal zero Then the equation can be solved by finding the values of x which make the value of the function equal to zero, that is, finding the values of x where the graph intersects the x axis

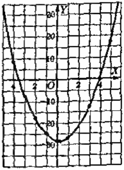
We shall now solve the equation graphically First we write the equation in the form $ax^2 + bx + c = 0$ It becomes $2x^2 - x - 28 = 0$ Then we let y equal the left member, thus $y = 2x^2 - x - 28$ Next we make a table for the equation $y = 2x^2 - x - 28$

$x =$	-4	-3	-2	-1	0	$\frac{1}{4}$	1	3	4	5
$y =$	8	-7	-18	-25	-28	$-28\frac{1}{8}$	-27	-13	0	17

Then we draw the graph of $y = 2x^2 - x - 28$, as shown on the next page The graph intersects the x axis at $x = 4$ and $x = -3\frac{1}{2}$ The roots of the equation are 4 and $-3\frac{1}{2}$ because the value of y and $2x^2 - x - 28$ at these points is zero

This work can be simplified by (1) noting that the curve is concave upward, (2) plotting the turning point, (3) drawing

the axis of symmetry, (4) plotting points to the right of the axis of symmetry, (5) plotting points of the curve to the left of the axis of symmetry (6) finding the points of intersection of the curve and the x axis



Any quadratic equation whose graph intersects the x axis has real roots and can be solved graphically If the graph of an equation does not intersect the x axis the equation cannot be solved graphically, and its roots are imaginary The equation $x^2 + x + 2 = 0$ has imaginary roots

To Solve a Quadratic Equation Graphically

1 Write the equation in the form $ax^2 + bx + c = 0$
2 Let y equal the left member of the equation found in step 1
3 Graph the equation obtained in step 2
4 Find the abscissas of the points where the graph intersects the x-axis. These are the roots of the equation.

EXERCISES

(A)

Solve graphically

1 $x^2 - x - 2 = 0$
2 $x^2 + x - 2 = 0$
3 $x^2 - 3x + 2 = 0$
4 $x^2 - 4 = 0$
5 $x^2 - 25 = 0$
6 $x^2 - 3x = 0$
7 $x^2 - 5x = -6$
8 $3x^2 + 5x = 2$
9 $2x^2 - 3x = 5$
10 $4x^2 - 20x + 25 = 0$

Find graphically the approximate roots of the following

11 $2x^2 - x = 1$
12 $3x^2 - 8x = 3$
13 $3x^2 - 2x = 0$
14 $x^2 - \frac{1}{4}x - \frac{3}{8} = 0$

(B)

15 On the same set of axes draw the graphs of the functions $x^2 - 2x - 3$ and $-x^2 + 2x + 3$ Where do the graphs intersect each other? What do you know about the roots of the equations $x^2 - 2x - 3 = 0$ and $-x^2 + 2x + 3 = 0$?

16 On the same set of axes draw the graphs of the functions $x^2 - 3x$ $x^2 - 3x + 3$ and $x^2 - 3x - 2$ Choose any value

of x. For this value of x, how much greater is $x^2 - 3x + 3$ than $x^2 - 3x$? than $x^2 - 3x - 2$? Choose any other value of x and answer the same questions about $x^2 - 3x + 3$.

17. Find the side of a square whose area is 90 square inches ($y = x^2 - 90$).

Algebraic Solution of Incomplete Quadratic Equations [A]

An incomplete quadratic equation is one having the form $ax^2 + b = 0$. It is sometimes called a pure quadratic equation because the unknown is only of the second degree. This type can be solved easily by the square-root method.

To Solve an Incomplete Quadratic Equation

1. Solve the equation for the square of the unknown number.
2. Find the square roots of both members of the equation.

Example 1 Solve $7x^2 - 175 = 0$.

Solution

$$7x^2 - 175 = 0$$
$$7x^2 = 175$$
$$x^2 = 25$$

R_2* $\quad x = \pm 5$

When we found the square roots of x^2 and 25, we might have written the results $\pm x = \pm 5$. Then we should have had $+x = +5$, $+x = -5$, $-x = +5$, and $-x = -5$. Since we are solving only for the value of $+x$, we need not consider the value of $-x$. Also, the last two equations are equivalent to the first two.

$-5y^2 = 12$

$y^2 = \frac{12}{-5}$

Example 2 Solve $3y^2 - 5 = 8y^2 + 7$.

Solution

$$3y^2 - 5 = 8y^2 + 7$$
$$3y^2 - 8y^2 = 7 + 5$$
$$-5y^2 = 12, \quad y^2 = -\tfrac{12}{5}$$
$$y = \pm\sqrt{\frac{-12}{5}} = \pm\frac{2i}{5}\sqrt{15}$$

PROOF Does $3\left(\pm\frac{2i\sqrt{15}}{5}\right)^2 - 5 = 8\left(\pm\frac{2i\sqrt{15}}{5}\right)^2 + 7$?

Does $-7\frac{1}{5} - 5 = -19\frac{1}{5} + 7$? Yes.

*R_2 means "Find the square roots of both members of the equation."

EXERCISES [A]

Solve the following equations, expressing any radical root in its simplest radical form

1. $t^2 - 49 = 0$
2. $4x^2 - 100 = 0$
3. $5y^2 - 20 = 0$
4. $4y^2 - 25 = 0$
5. $2y^2 - 36 = 0$
6. $4y^2 - 256 = 0$
7. $5x^2 - 40 = 0$
8. $\frac{x}{4} = \frac{9}{x}$
9. $3n^2 + 9 = 57$
10. $5x^2 - 55 = 90$
11. $9x^2 + 1 = 2$
12. $1 = \frac{1}{3}x^2$
13. $p^2 - 6 = 30$
14. $4x^2 - 147 = x^2$

$5t^2 - 40 = 0$ $t^2 - 8 = 0$ $5t^2 = 40$ $t^2 = 8$ $t =$

15. $3x^2 - 100 = -19$
16. $\frac{y}{5} = \frac{10}{y}$
17. $1.04r^2 = 66.56$
18. $x^2 = 151.23 - 2x^2$
19. $5y^2 + 500 = 844.45$
20. $\frac{x}{9} = \frac{36}{4x}$
21. $\frac{4}{x-3} = \frac{1}{3} + \frac{4}{x+3}$
22. $(2x-1)(3x+2) = x + 292$
23. $(x+4)(x-6) = 25 - 2x$
24. $\frac{4}{x^2 - 7} = 1$
25. Solve $x^2 = 9a^2$ for x, solve it for a
26. Solve $mx^2 = 1$ for x, solve it for m
27. Solve $V = \pi r^2 h$ for h
28. Solve $s = \frac{1}{2}gt^2$ for t

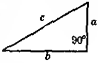
The Pythagorean Theorem [A]

If you have studied plane geometry, you are familiar with the Pythagorean Theorem

Pythagorean Theorem The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs

This is called the Pythagorean Theorem because Pythagoras, the great Greek mathematician is supposed to have given the first rigor-

ous proof of it This was about 525 B C The theorem may be expressed as a formula, where c is the hypotenuse and a and b the sides of a right triangle



$$c^2 = a^2 + b^2$$

Example. Find the length of a rectangle whose diagonal is 156 inches and whose width is 60 inches

Solution $x^2 + 60^2 = 156^2$

$x^2 + 3600 = 24{,}336$

$x^2 = 20{,}736$

$x = \pm 144$

64
4)256
24
16

The length is 144 inches

Since -144 has no meaning in this problem, we discard it

PROOF. $60^2 + 144^2 = 156^2$

[A] **PROBLEMS**

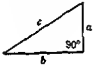
Find the hypotenuse c to the nearest tenth when

1 $a = 45$ and $b = 60$

2 $a = 28.8$ and $b = 12.0$

3. $a = 20$ and $b = 30$

4 $a = 40$ and $b = 75$



Find a to the nearest tenth when

5 $c = 156$ and $b = 144$

6 $c = 65.5$ and $b = 39.3$

7 $b = 18$ and $c = 20$

8 $b = 4\sqrt{2}$ and $c = 5\sqrt{2}$

9 Solve the formula $c^2 = a^2 + b^2$ for b, and make a rule for finding a leg of a right triangle when the hypotenuse and the other leg are given

10 Find the diagonal of a square whose side is 14 inches

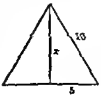
11 Find the side of a square whose diagonal is 50 inches

12. Show that the diagonal of a square whose side is s can be found by the formula $d = s\sqrt{2}$

13. Find the altitude of an equilateral triangle having a side of 10 inches

14. Find the altitude of an equilateral triangle having a side of 16 inches

15. Find the side of an equilateral triangle whose altitude is $8\sqrt{3}$ inches



16 Find the altitude of an equilateral triangle having a side s

17 Use the result of Exercise 16 to find the altitude of an equilateral triangle whose side is 20 inches

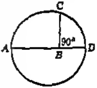
18 Find the diagonal of a rectangle whose dimensions are 18 feet and 24 feet

19 When the foot of a ladder 18 feet long is set 8 feet from the wall of a building, the top of the ladder just reaches the second story window How far from the ground is the window?

20 A baseball diamond is a square 90 feet on a side If the catcher, standing 5 feet behind home plate, makes a throw to second base, find the length of the horizontal path of the ball

21 Find the area of a right triangle if its hypotenuse is 52 inches long and one of its legs is 48 inches long

22 BC is the mean proportional between AB and BD →



a Find BC to the nearest tenth when $AB = 40$ and $BD = 20$

b. Find BC when $AB = 15\sqrt{2}$ and $BD = 3\frac{1}{3}\sqrt{2}$

Algebraic Solutions of Complete Quadratic Equations [A]

There are three algebraic methods of solving complete quadratic equations These methods are

1 Solution by factoring
2 Solution by completing the square It is a square-root method
3 Solution by formula

The last two of these methods can be used to solve any quadratic equation The first method is used only when the roots are rational and when $ax^2 + bx + c$ is easily factored For practical problems either of the last two methods is preferable

Completing the Square [A]

Before we attempt the algebraic solution of a quadratic equation by completing the square we should understand the meaning of "completing the square" and be able to complete the square quickly

At this time you should review squaring binomials (p 67) and finding square roots of trinomials (p 74)

We shall now learn how to make a perfect square trinomial when the first two terms are given You can learn the method by studying the following example

Example. Make $x^2 + 6x$ a perfect square trinomial by adding one term

Solution A square root of x^2 is x Then x is the first term of the square root of the trinomial Then $x^2 + 6x + ? = (x + ?)^2$

We know that $+6x = 2 \times x \times$ the missing number in the parenthesis

Then the missing number in the parenthesis is one half of $+6$, or $+3$

Now we can write $x^2 + 6x + ? = (x + 3)^2$ The missing term of $x^2 + 6x + ?$ is 3^2, or 9 Then the perfect square trinomial is $x^2 + 6x + 9$ We have completed the square

(A)

ORAL EXERCISES

What must be added to each of the following binomials to make them perfect square trinomials?

1. $x^2 + 8x + ?$	6 $x^2 + 7x + ?$	11. $x^2 + \frac{2}{3}x + ?$
2 $y^2 + 2x + ?$	7 $x^2 - 7x + ?$	12. $x^2 - \frac{4}{5}x + ?$
3 $m^2 - 6m + ?$	8. $c^2 - 3c + ?$	13. $y^2 + \frac{1}{3}y + ?$
4 $h^2 + 10h + ?$	9. $x^2 - 11x + ?$	14 $c^2 - c + ?$
5. $p^2 - 12p + ?$	10 $x^2 - 5x + ?$	15. $x^2 + 1x + ?$

Solving Complete Quadratic Equations by Completing the Square (A)

You solved an incomplete quadratic equation by solving for the square of the unknown and taking the square roots of both members of the equation

Let us now consider the complete quadratic equation $x^2 + 6x = 27$. We cannot take the square root of the left member, $x^2 + 6x$ If we write the equation in the form $x^2 = 27 - 6x$ and take the square root of both members, we get $x = \pm\sqrt{27 - 6x}$ We have not solved the equation for x because we have x occurring in the root

We can make the left member of $x^2 + 6x = 27$ a perfect square by adding $(\frac{1}{2} \text{ of } 6)^2$ to it A square root of $x^2 + 6x + 9$ is $x + 3$ If we add 9 to the right member, we must add 9 to the left member, obtaining $x^2 + 6x + 9 = 36$ Taking the square roots of both members, we get $x + 3 = \pm 6$

$$\begin{array}{r} x+3 \\ x+3 \\ \hline x^2+3x \qquad \\ 3x+9 \\ \hline x^2+6x+9 \end{array}$$

We have changed the quadratic equation $x^2+6x+9=36$ into the two linear equations $x+3=+6$ and $x+3=-6$. The roots of these two equations are 3 and -9.

The solution is written as follows

$$
\begin{aligned}
x^2+6x &= 27\\
x^2+6x+9 &= 36\\
x+3 &= \pm 6\\
x &= -3\pm 6\\
x &= 3 \text{ or } -9
\end{aligned}
$$

To Solve a Quadratic by Completing the Square

1. Change the equation to the form $x^2+bx=c$
2. Add to each member the square of half the coefficient of x
3. Find the square root of each member writing the double sign $\pm$ before the square root of the right member
4. Solve the resulting linear equations
5. Prove that the values of x found in step 4 satisfy the original equation

Example 1 Solve $x^2+6x-55=0$

Solution $\quad x^2+6x-55=0$

$\quad x^2+6x=55$

$A_{(\frac{1}{2}\cdot 6)^2}$ $\quad x^2+6x+9=64$

R_2 $\quad x+3=\pm 8$

$\quad x=-3\pm 8$

$\quad x=5 \quad -11$

PROOF Does $5^2+6(5)-55=0$? Yes

Does $(-11)^2+6(-11)-55=0$? Yes

EXERCISES

[Coefficient of x even] [A]

Solve each of the following quadratic equations by completing the square

1. $x^2+4x=-3$
2. $x^2-12x=-27$
3. $x^2+8x=0$
4. $y^2-2y=35$
5. $m^2+8m=-7$
6. $x^2-7=6x$
7. $c^2-8=2c$
8. $y^2+14y+48=0$
9. $x^2+8x=-15$
10. $y^2+4y=32$

11. $x^2 - 2x = 24$
12 $y^2 + 4y = 96$
13 $m^2 - 35 = 2m$
14 $x^2 - 15 - 2x = 0$
15 $p^2 + 2p - 48 = 0$
16 $h^2 + 2h = 63$

Example 2. Solve $y^2 - 5y = 24$

Solution This equation is like the equations above except that the coefficient of y is an odd number

$$y^2 - 5y = 24 \quad [\tfrac{1}{2} \text{ of } 5 = \tfrac{5}{2}, \text{ and } (\tfrac{5}{2})^2 = \tfrac{25}{4}]$$

$A_{\frac{25}{4}}$ $\quad y^2 - 5y + \frac{25}{4} = \frac{121}{4}$

R_2 $\quad y - \frac{5}{2} = \pm \frac{11}{2}$

Solving $\quad y = \frac{5}{2} \pm \frac{11}{2}$

$$y = 8 \text{ or } -3$$

The proof is left to the student

[Coefficient of x odd] [A]

EXERCISES

Solve

1 $x^2 + 11x = 12$
2 $x^2 - 3x = 4$
3 $c^2 - 5c - 14 = 0$
4 $y^2 - 3y - 54 = 0$
5. $y^2 - 5y = -4$
6 $h^2 + 3h = -2$
7. $k^2 + 5k - 14 = 0$
8. $x^2 - 9x + 8 = 0$
9 $w^2 + 3w = 28$
10 $x^2 + 5x - 36 = 0$
11. $x^2 + 3x = 54$
12. $p^2 + 7p + 6 = 0$
13 $x^2 + x = 12$
14. $y^2 + y = 56$
15 $k^2 - k = 6$
16. $w^2 - w = 42$

Example 3 Solve $3y^2 - 5y - 2 = 0$

Solution $\quad 3y^2 - 5y - 2 = 0$

$$3y^2 - 5y = 2$$

D_3 $\quad y^2 - \frac{5}{3}y = \frac{2}{3} \quad [\tfrac{1}{2} \text{ of } \tfrac{5}{3} = \tfrac{5}{6}, \text{ and } (\tfrac{5}{6})^2 = \tfrac{25}{36}]$

$A_{\frac{25}{36}}$ $\quad y^2 - \frac{5}{3}y + \frac{25}{36} = \frac{49}{36}$

R_2 $\quad y - \frac{5}{6} = \pm \frac{7}{6}$

$$y = \frac{5}{6} \pm \frac{7}{6}$$

$$y = 2 \text{ or } -\frac{1}{3}$$

PROOF Left to the student

$x = -\frac{2}{3} \pm \frac{\sqrt{7}}{3}$

$\pm \frac{\sqrt{7}}{\sqrt{9}} \quad \pm \frac{\sqrt{7}}{3}$

EXERCISES

[A]

Solve the following equations, expressing irrational roots in simplest radical form

1 $2x^2 - 22x = -28$

2 $x^2 + 36 = 13x$

3 $5x^2 + 16x + 6 = 0$

4 $6x^2 + 5x - 3 = 0$

5 $4x^2 + 11x = 3$

6 $-x^2 + 1 = x$

7 $6c - 2c^2 + 1 = 0$

8 $5y^2 + 2y - 2 = 0$

9 $-3h^2 + 5h - 2 = 0$

10 $2k^2 - 8k + 5 = 0$

Example 4 Express the roots of $3x^2 + 4x - 1 = 0$ to the nearest hundredth

Solution

$$3x^2 + 4x = 1$$
$$x^2 + \tfrac{4}{3}x = \tfrac{1}{3}$$
$$x^2 + \tfrac{4}{3}x + \tfrac{4}{9} = \tfrac{7}{9}$$
$$x + \tfrac{2}{3} = \pm \tfrac{1}{3}\sqrt{7}$$
$$x = \frac{-2 \pm \sqrt{7}}{3}$$
$$x = \frac{-2 \pm 2.646}{3}$$
$$x = 0.22 \text{ or } -1.55$$

The approximate value of $\sqrt{7}$ can be found in Table 1 at the end of the book or by the method on page 206

Notice that only the exact roots $x = \frac{-2 \pm \sqrt{7}}{3}$ will satisfy the equation

EXERCISES

[A]

Find the roots of the following to the nearest hundredth

1 $x^2 - 2x - 5 = 0$

2 $x^2 + 4x + 1 = 0$

3 $6x - x^2 = 7$

4 $y^2 - 12y + 33 = 0$

5 $y^2 + 23 = 10y$

6 $3m^2 - 2 = m$

7 $2x^2 - 60 = -x$

8 $y^2 = 3y + 3$

The Quadratic Formula [M]

We shall now show how to solve the general quadratic equation $ax^2 + bx + c = 0$

STEP 1
$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

STEP 2 $\frac{1}{2}$ of $\frac{b}{a} = \frac{b}{2a}$, and $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$

$A_{\frac{b^2}{4a^2}}$
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$= \frac{b^2 - 4ac}{4a^2}$$

STEP 3 R_2
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

STEP 4
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving Quadratic Equations by the Quadratic Formula[A]

The quadratic formula can be used to solve any quadratic equation

General quadratic equation $ax^2 + bx + c = 0$
Solution. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (quadratic formula)

To use the quadratic formula in the solution of a quadratic equation, it is only necessary to substitute in it the values of a, b, and c of the given equation See if you can learn the formula in three minutes

Example 1. Solve $3x^2 - 2x = 8$ by the formula

Solution We first write the equation in the form
$$ax^2 + bx + c = 0$$
obtaining
$$3x^2 - 2x - 8 = 0$$

Comparing the coefficients of these two equations, we find that $a = 3$, $b = -2$, and $c = -8$

The formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(cont on p 266)

Substituting the values of a, b, and c

$$x = \frac{+2 \pm \sqrt{4 - 4(3)(-8)}}{6}$$

Can you give a reason why the 3 and -8 are enclosed by parentheses?

$$x = \frac{2 \pm \sqrt{100}}{6}$$

$$x = \frac{2 \pm 10}{6}$$

$$x = 2 \text{ or } -\tfrac{4}{3}$$

The proof is left to the student. Prove that both 2 and $-\frac{4}{3}$ satisfy $3x^2 - 2x = 8$.

Example 2 Solve $3x^2 - 4x - 2 = 0$

Solution In this equation $a = 3$, $b = -4$, and $c = -2$

The formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substituting the values of a, b, and c

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{4 \pm \sqrt{40}}{6} = \frac{4 \pm 2\sqrt{10}}{6} = \frac{2 \pm \sqrt{10}}{3}$$

[A]

EXERCISES

Solve by the formula

1. $3x^2 + 8x + 5 = 0$
2. $2x^2 + 2x - 24 = 0$
3. $5x^2 + 26x + 5 = 0$
4. $y^2 + 6y = 55$
5. $8m^2 - 2m = 1$
6. $x^2 - 3x = 2$
7. $m^2 - 5 = 5m$
8. $x^2 + x - 2 = 0$
9. $2x^2 = 1 - 3x$
10. $3x = 2 - 9x^2$
11. $x^2 + x + 1 = 0$
12. $1 - 4x = 8x^2$
13. $x^2 + 6x + 9 = 0$
14. $x^2 - 81 = 0$
15. $3x^2 + 4x - 6 = 0$
16. $11x^2 - 12x = -3$

The Four Methods of Solving Quadratic Equations [A]

You have used four methods in solving quadratic equations. Any quadratic equation can be solved by completing the square or by the formula, which was made by completing the square.

Any quadratic equation which has real roots can be solved graphically, but the roots are only approximate, since they depend to some extent upon the sense of sight

An incomplete quadratic equation can always be solved by the square-root method Sometimes it can be solved by factoring and by graphing It can be solved by the formula, but this method is a waste of time

EXERCISES

[A]

Solve the following equations, using the method or methods designated by your teacher

1 $7x^2 - 11x = -4$	8 $x(x-2) = 7$
2 $y^2 + y - 1 = 0$	9 $3 - x(x-3) = 4$
3 $m^2 - 3m + 2 = 0$	10 $5x + x(x-7) = 9$
4 $x^2 + .7x = 12$	11 $(x-3)(x+4) = 3x - 5$
5 $6x^2 - 7x + .02 = 0$	12 $(x-3)^2 + 2(x+5)^2 = 40$
6 $\frac{m^2-3}{2} + \frac{m}{4} = 1$	13 $\frac{x-1}{2} - \frac{3x+1}{3} + \frac{4x^2}{3} = 0$
7. $x^2 + 100x = 4224$	14. $0.8x^2 - 3.8x = 1$

Finding Cube Roots[B]

Example. Find the three cube roots of 1

Solution
$$x^3 = 1$$
$$x^3 - 1 = 0$$

Factoring
$$(x-1)(x^2+x+1) = 0$$

$$\begin{array}{r|l} \text{If } x - 1 = 0 & \text{If } x^2 + x + 1 = 0 \\ x = 1 & x = \dfrac{-1 \pm i\sqrt{3}}{2} \end{array}$$

EXERCISES

1 Show that $\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1$

2. Find the three cube roots of 8

3 Find the three cube roots of -1

4 Find the three cube roots of 27

5 Find the three cube roots of 64.

Forming Equations When the Roots Are Given [A]

By reversing the process of finding the roots of an equation by factoring, an equation can be formed when the roots are given

Example Form the equation whose roots are 3 and $-\frac{1}{2}$

Solution.

$$\begin{array}{c|c} x = 3 & x = -\frac{1}{2} \\ x - 3 = 0 & 2x = -1 \\ & 2x + 1 = 0 \end{array}$$

$$(x-3)(2x+1) = 0$$
$$2x^2 - 5x - 3 = 0$$

EXERCISES

[A]

Form the quadratic equations whose roots are

1. 3 and -4
2. -1 and -3
3. 2 and -2
4. 3 and 0
5. 0 and -6
6. $\frac{3}{5}$ and -1
7. $\frac{2}{3}$ and -4
8. $-\frac{1}{3}$ and $\frac{1}{3}$
9. b and 0

[B]

Form the equations whose roots are

10. $\sqrt{3}$ and $\sqrt{5}$
11. $\sqrt{2}$ and $\sqrt{3}$
12. t and $2t$
13. 2, 1, and 3
14. -1, 2, and -5
15. 3, 0, and -4

Fractional Quadratic Equations [A]

Example Solve $\dfrac{x-4}{x-5} - \dfrac{2x-1}{x+4} = \dfrac{1-2x}{x^2-x-20}$

Solution

$$\frac{x-4}{x-5} - \frac{2x-1}{x+4} = \frac{1-2x}{x^2-x-20}$$

$$\frac{x-4}{x-5} - \frac{2x-1}{x+4} = \frac{1-2x}{(x-5)(x+4)}$$

$M_{(x-5)(x+4)}$,

$$(x+4)(x-4) - (x-5)(2x-1) = 1 - 2x$$
$$(x^2-16) - (2x^2 - 11x + 5) = 1 - 2x$$
$$x^2 - 16 - 2x^2 + 11x - 5 = 1 - 2x$$
$$-x^2 + 13x - 22 = 0$$

By the formula, $x = \dfrac{-13 \pm \sqrt{169 - 4(-1)(-22)}}{-2}$

$$x = 2 \text{ or } 11$$

PROOF For $x = 2$

Does $\dfrac{2-4}{2-5} - \dfrac{4-1}{2+4} = \dfrac{1-4}{4-2-20}$?

Does $\frac{2}{3} - \frac{1}{2} = \frac{-3}{18}$?

Does $-\frac{3}{18} = -\frac{3}{18}$? Yes

For $x = 11$

Does $\frac{11-4}{11-5} - \frac{22-1}{11+4} = \frac{1-22}{121-11-20}$?

Does $\frac{7}{6} - \frac{7}{5} = -\frac{7}{30}$?

Does $-\frac{7}{30} = -\frac{7}{30}$? Yes

[A] **EXERCISES**

Solve the following equations

1 $\frac{x-1}{2x} = \frac{2}{x+3}$

2 $\frac{x}{3} + \frac{3}{x} = \frac{5}{2}$

3 $x + \frac{5}{2x} + \frac{7}{2} = 0$

4 $\frac{2x+1}{1-2x} = \frac{5x-38}{14}$

5 $\frac{x-1}{2} - \frac{5}{2} = \frac{2}{1-x}$

6 $\frac{8}{x-1} - \frac{4}{3x+1} = 1$

7 $\frac{2y-4}{3y-1} = y + 2$

8 $\frac{4x-10}{x+5} - \frac{7-3x}{x} = \frac{7}{2}$

9 $\frac{3}{x-2} - \frac{5}{x+2} = 4$

10 $\frac{3}{x-6} - \frac{1}{2} = \frac{2}{x-5}$

11 $\frac{5x+1}{2} = \frac{3}{2-x}$

12 $\frac{2x-1}{x-1} + \frac{x}{x+1} = \frac{1}{x^2-1}$

13 $\frac{3x-5}{9x} - \frac{1}{3} = x$

14 $\frac{2}{x-1} - \frac{3}{x-2} = \frac{2}{x+4}$

15 $\frac{2m}{m-1} - \frac{m+3}{m+1} = \frac{19}{m^2-1}$

16 $\frac{3}{x^2-1} + \frac{1}{2(x-1)} = \frac{1}{4}$

Literal Quadratic Equations[A]

A literal equation is one in which at least one of the known numbers is represented by a letter

A literal quadratic equation can always be solved by completing the square and by the formula Some can be solved by factoring

Example Solve $x^2 - 6bx - 7b^2 = 0$ for x

1 Solution by factoring

$$x^2 - 6bx - 7b^2 = 0$$
$$(x - 7b)(x + b) = 0$$

If $x - 7b = 0$, $x = 7b$ | If $x + b = 0$, $x = -b$

2 **Solution by completing the square.**

$$x^2 - 6bx - 7b^2 = 0$$

$$x^2 - 6bx = 7b^2$$

$A_{(\frac{1}{2} \text{ of } 6b)^2}$ $$x^2 - 6bx + 9b^2 = 16b^2$$

R_2 $$x - 3b = \pm 4b$$

$$x = 3b \pm 4b$$

$$x = 7b \text{ or } -b$$

3 **Solution by formula.**

$$x^2 - 6bx - 7b^2 = 0$$

$a = 1$, b in the formula $= -6b$ in the example, and $c = -7b^2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6b \pm \sqrt{36b^2 - 4(1)(-7b^2)}}{2}$$

$$x = \frac{6b \pm 8b}{2}$$

$$x = 7b \text{ or } -b$$

The proof is left to the student

EXERCISES

[A]

Solve for x

1. $x^2 + 3ax + 2a^2 = 0$	9. $2x^2 + 4bx - c = 0$
2. $x^2 + 6kx = 7k^2$	10 $x^2 + cx + d = 0$
3. $x^2 - 2cx = 35c^2$	11. $ax^2 + x + 1 = 0$
4. $x^2 - c^2 = 0$	12. $x^2 + 5x = 9a^2 + 15a$
5 $a^2x^2 - y^2 = 0$	13. $ax^2 + 2bx - 1 = 0$
6. $3x^2 - 2ax = 3a^2$	14. $\frac{1}{4}x^2 + \frac{1}{2}mx = \frac{1}{4}$
7 $2x^2 + x - c = 0$	15. $dx^2 - ex - f = 0$
8. $x^2 + 2cx + c^2 = 0$	16 $(1-a)x^2 - (1+a)x + a(a^2 - 1) = 0$

[B]

17. Solve $S = 2\pi r^2 + 2\pi rh$ for r
18. Solve $V = \frac{1}{3}\pi h(r^2 + R^2 + rR)$ for R
19. Solve $(x-1)^{-1} + (x+3)2^{-1} = 4\frac{1}{4}$

Radical Equations [A]

The radical equations which you solved when studying Chapter 8 reduced to linear equations We shall now solve radical equations which reduce to quadratic equations

Example Solve $3\sqrt{x^2+5} - 4x = 1$

Solution
$$3\sqrt{x^2+5} - 4x = 1$$
$$3\sqrt{x^2+5} = 4x + 1$$

P_2
$$9x^2 + 45 = 16x^2 + 8x + 1$$

Simplifying $7x^2 + 8x - 44 = 0$

Solving $x = 2$ or $-3\frac{1}{7}$

PROOF Does $3\sqrt{2^2+5} - 4(2) = 1$? Yes

Does $3\sqrt{(-3\frac{1}{7})^2+5} - 4(-3\frac{1}{7}) = 1$? No

Then only 2 is a root of the equation

Each time we square both members of an equation containing radicals, we must decrease the number of radicals by at least one All solutions of radical equations must be proved to eliminate the roots of any derived redundant equations which are not roots of the original equations

EXERCISES

[A]

Solve the following radical equations

1. $1 - x = \sqrt{x+5}$
2. $x - 1 = \sqrt{x-1}$
3. $x = 8 - \sqrt{x+4}$
4. $\sqrt{3x-2} = 5x - 8$
5. $\sqrt{x+4} = x - 8$
6. $2x = 3 + \sqrt{7x-3}$
7. $\sqrt{x+1} = \sqrt{4x+17} + 4$
8. $\sqrt{2x-1} = 6 - \sqrt{x+4}$
9. $\sqrt{y+4} + 20 = 2y$
10. $\sqrt{x+3} - \sqrt{x-2} = 1$
11. $\sqrt{2x+2} - \sqrt{x+9} = 0$
12. $2\sqrt{x-3} - 2 = \sqrt{x+4}$

[B]

Solve

13. $\sqrt{x+3} = \dfrac{8}{\sqrt{x-9}}$
14. $\dfrac{12}{\sqrt{5x+6}} = \sqrt{2x+5}$
15. $2\sqrt{y-1} - \sqrt{3y-5} = \sqrt{y-9}$
16. $\sqrt{x+1} - \sqrt{2x-5} = \sqrt{x-2}$

Problems That Lead to Quadratic Equations [A]

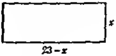
If you are solving a problem which is expressed by a quadratic equation, you will obtain two answers for the number represented by the letter in the equation Consequently you will usually have two answers for any other unknown quantity in the problem

Remember to use only one unknown letter in each problem, as we have not yet learned how to solve quadratic equations in two unknowns

Example The perimeter of a rectangle is 46 inches and the area is 120 square inches Find the dimensions of the rectangle

Solution Let $x =$ the number of inches in the width
Then $23 - x =$ the number of inches in the length

$$x(23 - x) = 120$$
$$23x - x^2 = 120$$
$$-x^2 + 23x - 120 = 0$$
$$x^2 - 23x + 120 = 0$$



$$x = \frac{23 \pm \sqrt{23^2 - 4(1)(120)}}{2}$$

$$x = 15 \text{ or } 8,\ 23 - x = 8 \text{ or } 15$$

The width is 15 inches and the length 8 inches or the width is 8 inches and the length 15 inches

In some problems a set of answers can be discarded For example, the dimensions of a rectangle are discarded when they are negative

NUMBER PROBLEMS

[A]

1 Separate 16 into two parts whose product is 63

2 If six times a number is subtracted from its square, the remainder is 891 What is the number?

3 If four times a number is added to three times its square, the sum is 95 Find the number

4 If twice a number is subtracted from eight times its square, the remainder is 1 Find the number

5 Find two consecutive integers whose product is 1722

6 A number increased by twice its square equals 300 What is the number?

7 If twice the square of a number is decreased by five times itself, the remainder is 150 Find the number

8 One number is 3 more than twice another, and the sum of their squares is 137 Find the numbers

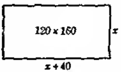
9 One number is 4 less than three times another Find the numbers if the difference of their squares is 160

10 One number exceeds another by 6 If the larger is divided by the smaller, the quotient is the same as when the smaller is divided by 25 Find the numbers

[A]

GEOMETRY PROBLEMS

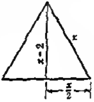
1. A rectangular field contains 120 acres Find its dimensions if its length exceeds its width by 40 rods (1 acre = 160 square rods)



2. The hypotenuse of a right triangle is 117 inches and one leg exceeds twice the other by 18 inches How long is each leg?

3. One leg of a right triangle is 105 inches Find the hypotenuse and the other leg if the hypotenuse exceeds twice this other leg by 7 inches

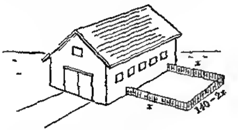
4 Find to the nearest tenth of an inch the side of an equilateral triangle if it exceeds the altitude by 2 inches

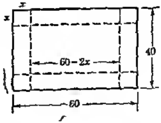


5 A man uses 100 feet of fencing to enclose a rectangular garden Find the dimensions of the garden if its area is a maximum

$A = x(50 - x)$ Complete the solution by finding the maximum value of the function

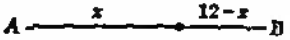
6. Find the dimensions of the largest rectangular yard that can be made using 140 feet of fencing for three sides of the yard and a side of a barn for the fourth side (see page 250)





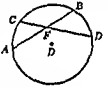
7. How wide a strip must be mowed around a rectangular grass plot 40 feet wide and 60 feet long for half of the mowing to be done? See diagram at right above

8 A rectangular wheat field is 160 rods long and 40 rods wide. How wide a border must be cut around the field for one third of the wheat to be cut?



9 A line is divided into extreme and mean ratio when the larger segment is the mean proportional between the whole line and the smaller segment. Find the lengths of the segments when a 12-inch line segment is divided into extreme and mean ratio.

10 A line segment 16 inches long is divided into extreme and mean ratio. Find the lengths of the two segments into which the line segment is divided.

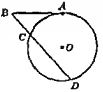


11 AB and CD are two chords of a circle intersecting at F. It is proved in geometry that $AF \cdot FB = CF \cdot FD$.

a Find CF and FD if $CD = 30$, $AF = 8$, and $FB = 18$.

b Find AF and FB if $CF = 10$, $FD = 20$, and $AB = 33$.

12 It is proved in geometry that the tangent BA is the mean proportional between the secant BD and the external segment BC.



a Find AB if $BC = 4$ inches and $CD = 16.25$ inches.

b Find BC and CD when $AB = 16$ inches and $BD = 40$ inches.

c If $CD = 10$ inches and $BA = 18$ inches, find BC to the nearest tenth of an inch.

MOTION PROBLEMS

[A]

1 The time going 180 miles was 1 hour more than the time returning. Find the rate in each direction if the rate returning was 15 miles an hour faster than the rate going.

2 The highway distance between two cities is 280 miles. The rate for 80 miles of the trip from one city to the other is to be 10 miles an hour faster than the rate for the remainder of

the distance Find the two rates if the time of the trip is 6 hours

3. A plane flies 300 miles having a tail wind of 10 m p h, and returns against a wind of 20 m p h Find the speed of the plane in still air if the time of the flight is $4\frac{1}{2}$ hours

4. At what air speed must a plane be flown to complete a trip between two airports 420 miles apart in 5 hours if the flight going has a head wind (blowing against the nose of the plane) of 40 miles an hour and the flight returning has a tail wind of 30 miles an hour?

5. A plane goes 150 miles with the wind and returns against the wind in 3 hours and 45 minutes Find the air speed of the plane if the wind blows 10 miles an hour

6 It takes a crew 1 hour and 30 minutes longer to go 12 miles up a river than to return Find the rate of the crew in still water if the rate of the current is 2 miles an hour

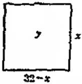
7 A speed boat has a speed of 27 miles an hour in still water If it requires $1\frac{1}{8}$ hours for the boat to go 15 miles downstream and return, what is the rate of the current?

8. An airplane pilot has 1 hour and 10 minutes to go 100 miles and return When going he has a tail wind of 20 miles an hour, and returning he has a head wind of 20 miles an hour He estimates he will lose $2\frac{1}{2}$ minutes of flying time in take-offs and landings At what air speed must he fly?

MAXIMA AND MINIMA PROBLEMS

(B)

1. Mrs Lane had 64 yards of wire fencing to enclose a chicken yard She asked a mathematics teacher to find the dimensions of the yard which would be rectangular and have the greatest area Show that the yard would be a square with each side 16 yards long



Hint $y = x(32 - x)$ Find the value of x that makes y a maximum

2. The sum of two numbers is 136 Find the numbers if their product is a maximum (Let $x =$ one of the numbers and let $y =$ the product)

3 I am thinking of a number I subtract 8 from the number, and then multiply the difference by the number Find the number of which I am thinking if the product is the smallest possible

4 A boy shoots an arrow vertically upward in the air Its height in feet after t seconds is given by the formula $h = 100t - 16t^2$

a Graph the equation

b Find the time that the arrow is in the air ($h = 0$)

c Find the greatest height of the arrow

Checking Your Understanding of Chapter 9

Before you begin the chapter review exercises be sure that you know

	PAGE
1 The meaning of a quadratic function and of a quadratic equation	247
2 How to graph a quadratic function	247
3 How to find the turning point of the graph of a quadratic function	248
4 How to find the maximum and minimum values of quadratic functions	250
5 How to find the intercepts of the graphs of quadratic functions	249
6 How to solve incomplete quadratic equations	
(*a*) by graphing	255
(*b*) by the square-root method	257
7 How to solve complete quadratic equations	
(*a*) by factoring	90
(*b*) by graphing	255
(*c*) by completing the square	261
(*d*) by formula	265

8 How to spell and use correctly these words

MATHEMATICAL VOCABULARY

axis of symmetry (p 251)	intercept (p 249)
complete quadratic (p 254)	maximum (p 250)
incomplete quadratic (p 254)	minimum (p 250)

[A] CHAPTER REVIEW

1. What is the name of the graph of $y = 2x^2 - x + 1$?

Form the equations whose roots are

2. -3 and 6

3. $\frac{1}{2}$ and $\frac{1}{3}$

Solve by factoring:

4. $x^2 + 2 = 3x$

5. $2x^2 + 9x = 5$

Solve by completing the square

6. $x^2 + 2x = 48$

7. $x^2 + 6x = 91$

8. $x^2 + x - 5 = 0$

9. $3x^2 + 6x - 45 = 0$

Solve by the formula

10. $8y^2 - 2y - 1 = 0$

11. $2m^2 - 3m = 2$

12. $3x^2 + 5x = 2$

13. $2y^2 - 10y = -9$

14. Find the maximum value of the function $6x - x^2$

15. Solve graphically $x^2 - 2x = 15$

16. One number exceeds another by 8. The square of the larger number exceeds the square of the smaller by 208. Find the numbers

17. Find two consecutive odd numbers whose product is 1295

18. The hypotenuse of a right triangle is 50 inches. One leg of the triangle exceeds the other by 10 inches. Find the two legs

19. Which two of these functions have graphs that can be made to coincide: $2x^2 - x + 3$, $2x^2 - x + 6$, and $x^2 - 2x + 3$?

Solve by any method

20. $\frac{1}{x-4} - \frac{14}{x+2} + 1 = 0$

21. $\frac{3}{x} + \frac{x}{3} = 2\frac{1}{2}$

22. The length of a rectangle is 1 foot greater than twice its width. Find the dimensions of the rectangle if its area is 78 square feet

[A] CUMULATIVE REVIEW

1. Simplify $3a - [a - 6(2a + 1)]$

2. Find the value of $(64)^{-\frac{1}{2}}$

3. Multiply $(3 - 2i)(3 + i)$

4. Solve and prove $2\sqrt{x-3}-6=0$

5. Solve $S=e^2\sqrt{3}$ for e

6 Find the square roots of 93 1225

7. $\frac{4}{5}$ is what per cent of $1\frac{1}{4}$?

8. What number, increased by 25% of itself, equals 80?

9. Factor x^2-a^2+6a-9

10. $f(x)=x^3+2x^2+10$ Find $f(-4)$

11 Complete The graph of $y=x^2+2x+1$ is a --?--

12 Find the side of an equilateral triangle whose altitude is 12 $5\sqrt{3}$ inches

13. Form the equation whose roots are $\frac{2}{3}$ and $-\frac{1}{2}$

14. How much water must be added to 10 gallons of a 3% salt solution to make it a 2% solution?

15 Solve $\frac{2}{x-1}-\frac{3}{2x+5}=\frac{5}{3}$.

16 Simplify the following radicals

a $\sqrt{12}$ b. $\sqrt{96}$ c $\sqrt{\frac{2}{3}}$ d. $\sqrt{\frac{a}{b}}$ e. $\sqrt{5a^2b^3}$ f $\sqrt{\frac{x^2}{y^3}}$

17. Solve $6x^2-x+1=0$

18. Solve for x $x^2+6a^2=5ax$

19. Find to the nearest hundredth the value of $\frac{3-2\sqrt{3}}{5}$.

20 Rationalize the denominator of $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

21. Simplify $(a-b)^2-(a+2b)^2$

22. Simplify $\frac{\sqrt{12}}{\sqrt{-6}}$, $3\sqrt{-2}\times 4\sqrt{-50}$, $\sqrt[3]{8^2}$

23. The average rate of an automobile between two cities 180 miles apart was 5 miles an hour more than that of a train If the automobile made the trip in 30 minutes less time than the train, find the rate of each

24. Divide x^3-3x^2+3x-1 by $x-1$

25 Expand $(a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2$

26. Multiply $x^n(x^{2n}-3x^{1-n})$

27 Find three consecutive even integers such that the square of the largest will exceed the product of the other two by 88

28 Solve for x $m^2(x-1) = 1 + m(x-2)$

29 Solve for R $W = P\dfrac{2R}{R-r}$

30 Solve $2x^{-\frac{1}{2}} = 6$

31 Solve $\dfrac{x+2}{x-4} + \dfrac{x^2+22}{x^2-7x+12} = \dfrac{2x+3}{x-3}$

32 Find the minimum value of y if $y = 3x^2 - 4x + 2$

33 What is the slope of the graph of $2y = 3x - 1$?

34 Find the equation of the line whose slope is $-\frac{2}{3}$ and whose y-intercept is 4

35 x varies directly as y and x is 10 when y is 18 Find x when y is 30

36 x varies inversely as y If $x = 25$ when $y = 24$ find x when $y = 15$

37 Find the side of a square whose diagonal is $8\sqrt{2}$

38 Solve $x^{\frac{3}{5}} = -27$

39 Solve

$$\begin{aligned} x + y + z &= 4 \\ 2x + 3y - z &= 1 \\ 3x - y + 2z &= 1 \end{aligned}$$

40 C men can complete a job in d days How many more men should be hired to complete the job in x days?

41 The total income from two investments is \$317 50 The investment at 5% produces \$107 50 more than the investment at 3% How much is invested at each rate?

42 An observer in a balloon 2400 feet above town A finds that the angle of depression of town B is $21° 4'$ What is the distance between the towns if each has the same elevation above sea level?

43 How can you tell from the graph of a quadratic equation whether the roots of the equation are real?

After each of the following exercises four possible solutions (*a*), (*b*), (*c*), and (*d*) are given. Solve the exercises to find which solution is correct.

44. Find the quotient $\left(\frac{1}{x}+1\right) \div \left(1-\frac{1}{x^2}\right)$.

a. $\frac{x}{x-1}$ c. $1-\frac{1}{x}$

b. $\frac{x^3+x^2-x-1}{x^3}$ d. $\frac{1}{x}+1$

45. Solve $\frac{12}{\sqrt{3}}x + 15\,x\sqrt{\frac{1}{3}} = 243^{\frac{3}{5}}$

a. 1 c. $\sqrt{\frac{1}{3}}$

b. $\sqrt{3}$ d. $-\sqrt{3}$

46. Factor $x^4 + x^2y^2 + y^4$

a. $(x^2 - xy - y^2)(x^2 + xy + y^2)$

b. $(x^2 - xy + y^2)(x^2 + xy - y^2)$

c. $(x^2 - xy + y^2)(x^2 + xy + y^2)$

d. $(x^2 + xy + y^2)(x^2 + xy + y^2)$

47. Multiply $(2 - 3\,i)$ by $(4 + i)$

a. $5 - 14\,i$ c. $5 - 10\,i$

b. $11 - 10\,i$ d. $11 + 10\,i$

48. Simplify $3^{-1} + 4^{-1}$

a. $\frac{1}{12}$ c. $\frac{7}{12}$

b. $\frac{1}{7}$ d. 7^{-1}

49. Solve $(x-2)(x^2+2\,x+4)+x^3 = 2\,x^3 - 4\,x$

a. 1 c. 2

b. $1\frac{1}{2}$ d. 4

50. Write the equation passing through the point (3, 4) and parallel to the line $x - 2\,y = 0$

a. $x - 2\,y + 5 = 0$ c. $x + 2\,y - 5 = 0$

b. $x - 3\,y + 4 = \frac{1}{2}$ d. $3\,x - 4\,y = \frac{1}{2}$

51. The minimum value of $8\,x^2 - 15$ is

a. 8 c. -15

b. 0 d. $1\frac{7}{8}$

[A]

CHAPTER TEST

In this test express all irrational roots in their simplest radical forms

Part I

Solve by factoring

1 $x^2 - 36 = 0$

2. $x^2 + 9x = 0$

3 $x^2 + 7x = 18$

4 $4x^2 - 196 = 0$

Solve by completing the square

5 $x^2 + 2x - 1 = 0$

6. $x^2 + 10x = 75$

7. $y^2 - 5y - 3 = 0$

8 $2x^2 + 9x = 5$

Solve Exs 9–12 by formula

9 $x^2 - x = 56$

10. $3x^2 + 11x = 4$

11 $2x^2 + 5x = 12$

12 $x^2 + 2x + 5 = 0$

13. Solve $A = \pi r^2$ for r

14 Find the maximum value of $10 - 2x - x^2$

15 Draw the graph of the function $x^2 - 2x - 8$

Part II Problems

1. Find two consecutive integers whose product is 9702

2 The area of a rectangle is 154 square inches and the perimeter is 50 inches Find the dimensions of the rectangle

3. The square of a certain number is 45 more than 4 times the number Find the number

4 The sum of a certain number and five times its square is 994 Find the number

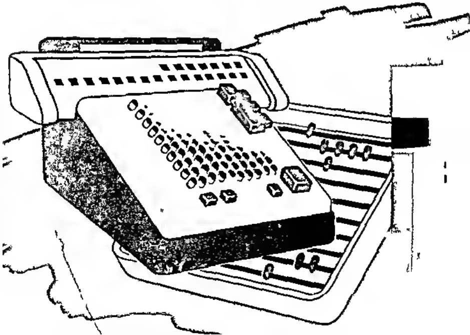
5 The diagonal of a rectangle is 100 inches and one leg is 30 inches Find the length of the other leg to the nearest tenth of an inch

6 One automobile travels 240 miles in 1 hour less time than a second automobile by going 12 miles an hour faster Find the rate of the slower automobile

Computing machines

The first adding machine was the abacus, which came into use in very early times and which is still used in some parts of the Orient In 1642 Blaise Pascal invented an adding machine which was a great improvement over earlier types of machines. In his machine one complete revolution of one wheel caused the wheel of the next higher order to make one tenth of a revolution. Nearly all calculating machines of today use this same principle

The first large scale digital calculating machine was the Automatic Sequence Controlled Calculator, or Mark I. The Mark I calculator was originated in 1937 by Howard H Aiken, Professor of Mathematics at Harvard University The designing and construction of the machine were carried out from 1939 to 1944 by the International Business Machines Corporation under the direction of Professor Aiken, B M. Durfee, F E. Hamilton, and C. D Lake Since then the International Business Machines Corporation and other companies, such as the Bell Laboratories and the Moore School of Electrical Engineering, have been making improved calculators



One of the newest high speed electronic calculators is the "704," built by the International Business Machines Corporation.

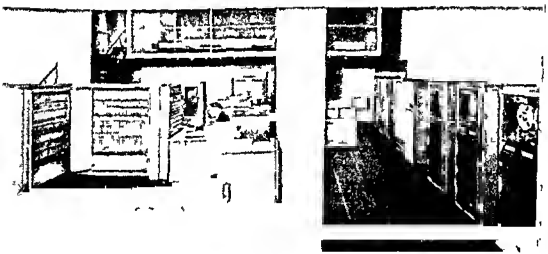
The 704 is capable of performing more than 40,000 addition or subtraction operations a second, and more than 5,000 multiplications or divisions of 10 digit numbers. In solving a typical problem, the 704 performs an average of 14,000 mathematical operations a second.

Internally, the 704 performs operations in the binary number system (see page 320). All initial data and final results may be in the familiar decimal number system. High speed conversion between number systems is handled automatically by the calculator.

The need for an electronic machine which will carry out thousands of operations a second is illustrated by the fact that the solution of a well known partial differential equation useful in aircraft wing design requires 8,000,000 calculating steps per case. The 704 completes the solution in a few minutes. A man working with a desk computer and using the same method would require seven years.

The electronic calculator 704

I.B.M.



Examples Rounded off to tenths,

3 24 becomes 3 2	8 783 becomes 8 8
4 65 becomes 4 6	19 05 becomes 19 0
7 35 becomes 7 4	23 146 becomes 23 1

ORAL EXERCISES [A]

Round off each of the following numbers to the nearest tenth

1 2 56	3 9 67	5 18 99	7. 7 65
2 18 41	4 5 971	6 4 35	8 20 750

Round off each of the following numbers to the nearest hundredth

9 7 861	11 10 158	13 2 135	15 8 1457
10 3 146	12 2 1963	14 4 245	16 9 350

Units of Measure [A]

The unit of measure in any measurement is the smallest unit used in the measurement Thus one foot is the unit of measure in the recorded measurement 3 yd 2 ft, and $\frac{1}{4}$ of an inch is the unit of measure in the recorded measurement 1 ft $2\frac{1}{4}$ in

Significant Figures [A]

In general, all the figures of a number are significant The digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 are *always* significant In some cases zero is significant and in some cases it is not significant Zero is not sig nificant if it is used merely as a place holder following a decimal point It is significant if it is used to show the precision of a measurement

For example, the zero in the recorded measurement 7 0 inches is significant because it shows that the measurement was made to the nearest tenth of an inch

The position of the decimal point has no bearing on the number of significant figures but it indicates the unit of measure and the precision of the measurement

The following rule will be useful in determining the significant figures of a number

Rule for Significant Figures

1 Disregard all initial zeros
2 Disregard all final zeros unless they follow a decimal point
3 The remaining digits are significant

[A]

EXERCISES

State the number of significant figures in each of the following

1. 4718	4 1810	7. 280 3	10 7 0
2 320	5 304 0	8 9000	11 21 04
3. 17 41	6 03 7	9 0 007	12 00 114

Round off each of the following numbers to three significant figures

13 714 2	15 1725	17. 0 1432	19 721 0
14 78 64	16 9 135	18 9147 2	20 46120

LOGARITHMS

We live in a machine age By using machines we do more work and do it in less time, thereby having more leisure time and enjoying a higher standard of living

To shorten the work in computation mathematicians have invented logarithms, the slide rule, and computing machines, and have developed new methods of solutions In this chapter we shall study logarithms and the slide rule

We shall first learn how to multiply, divide, and find powers and roots of numbers with logarithms, and then we shall learn how to perform these operations with the slide rule

Logarithms as a Time-Saver [A]

In order that you may appreciate the use of logarithms, we shall solve a problem in two ways—first without logarithms, and second with logarithms See how logarithms shorten paper work

Example Find the value of $\sqrt{\dfrac{7.52 \times 16.3}{28}}$

Solution 1 (without logarithms)

```
        7.52
        16.3
       2 256
      45 12
      75 2
     122.576

        4.3777
   28)122.5760
      112
       10 5
        8 4
        2 17
        1 96
          216
          196
           200
           196

   2. 0  9
  √4.37 77
   4
40 | 37
   | 00
409| 37 77
   | 36 81
        96    Answer 2.09
```

Solution 2 (with logarithms)

Let $x = \sqrt{\dfrac{7.52 \times 16.3}{28}}$

Then $x^2 = \dfrac{7.52 \times 16.3}{28}$

$$
\begin{aligned}
\log 7.52 &= 0.8762 \\
\log 16.3 &= 1.2122 \\
\log \text{product} &= 2.0884 \\
\log 28 &= 1.4472 \\
\log x^2 &= .6412 \\
\log x &= .3206 \\
x &= 2.09
\end{aligned}
$$

Meaning of Logarithms [A]

You may have heard that logarithms are exponents. Let us try to obtain a clearer idea of their meaning. In the statement $10^3 = 1000$, the 3 is the exponent of the base 10. If we use the word *logarithm* we say, "The logarithm of 1000 to the base 10 is 3." Again, from the statement $2^3 = 8$ we say that the exponent of 2 is 3 and that the logarithm (abbreviated *log*) of 8 to the base 2 is 3.

Other examples showing the meaning of logarithms are

$\log_{100} 10000 = 2$ means the same as $100^2 = 10000$
$\log_2 16 = 4$ means the same as $2^4 = 16$
$\log_9 81 = 2$ means the same as $9^2 = 81$

You should always remember that a logarithm is an exponent

The logarithm of a number to a given base is the exponent that indicates the power to which the base must be raised to equal that number

The expression $2^5 = 32$ is written in exponential notation and the equivalent expression $\log_2 32 = 5$ is written in logarithmic notation You must know how to change from either of these forms to the other

Example 1 Change $3^4 = 81$ to logarithmic notation

Solution The base is 3 and the logarithm is 4 Then $\log_3 81 = 4$

Example 2 Express $2^5 = 32$ in logarithmic form

Solution $\log_2 32 = 5$

Example 3 Express $\log_{10} 1000 = 3$ in exponential form

Solution $10^3 = 1000$

Example 4 Find $\log_4 64$

Solution $64 = 4^3$ Then $\log_4 64 = 3$

Example 5 Find $\log_{10} 0.0001$

Solution $0.0001 = \frac{1}{10000} = \frac{1}{10^4} = 10^{-4}$ Then $\log_{10} 0.0001 = -4$

(A)

Change the following expressions from exponential notation to logarithmic notation

1. $2^4 = 16$
2. $2^1 = 2$
3. $2^3 = 8$
4. $2^5 = 32$
5. $2^6 = 64$
6. $2^0 = 1$
7. $2^{-1} = \frac{1}{2}$
8. $2^{-3} = \frac{1}{8}$
9. $5^2 = 25$
10. $5^3 = 125$
11. $6^3 = 216$
12. $7^2 = 49$
13. $5^0 = 1$
14. $5^{-1} = .2$
15. $5^{-2} = 0.04$
16. $5^{-3} = 0.008$
17. $10^4 = 10,000$
18. $10^3 = 1000$
19. $10^2 = 100$
20. $10 = 10$
21. $10^{-1} = 0.1$
22. $10^{-2} = 0.01$
23. $10^5 = 100,000$
24. $10^{-3} = 0.001$

Change the following expressions to exponential notation:

25. $\log_{10} 1000 = 3$
26. $\log_{10} 10,000 = 4$
27. $\log_2 8 = 3$
28. $\log_7 49 = 2$
29. $\log_{10} 0.01 = -2$
30. $\log_{16} 256 = 2$
31. $\log_4 64 = 3$
32. $\log_9 81 = 2$
33. $\log_3 81 = 4$

Find the following logarithms

34 $\log_2 4$	38 $\log_4 256$	42 $\log_4 \frac{1}{8}$
35 $\log_{10} 100$	39 $\log_2 32$	43 $\log_{10} 0.1$
36 $\log_{10} 10000$	40 $\log_8 512$	44 $\log_{10} 0.01$
37 $\log_5 125$	41 $\log_3 27$	45 $\log_{10} 0.001$

46 Find $\log 4^2$ to base 4   47 Find $\log 10^3$ to base 10

Common Logarithms [A]

Any positive number except 1 can be used as the base of logarithms, but only two bases are commonly used

The natural system of logarithms has the base e, which has the approximate value 2.71828 The base e is useful in theoretical mathematics because the rate of change of the function e^x is equal to the value of e^x Natural logarithms are called Napierian logarithms in honor of John Napier, their inventor The expression $\log_e 45$ is often written in the abbreviated form $\ln 45$

Logarithms having 10 as the base are called common logarithms The base 10 is used in computation because we use the decimal system of notation Unless stated otherwise "logarithm" will mean "common logarithm" and it will not be necessary to write the base 10 For example, $\log 75$ means $\log_{10} 75$ Briggs (1560–1630), an English mathematician, by 1616 had completed a table of logarithms to the base 10 of all integers from 1 to 999 inclusive In other words he had prepared a table in which all integers from 1 to 999 inclusive were expressed as powers of 10 Later he extended the table to include all integers from 1 to 20,000

Decimal Powers of 10 [A]

The table shown here gives the integral powers of 10 from -3 to $+4$ inclusive From the table we may infer the following

$10000 = 10^4$
$1000 = 10^3$
$100 = 10^2$
$10 = 10^1$
$1 = 10^0$
$0.1 = 10^{-1}$
$0.01 = 10^{-2}$
$0.001 = 10^{-3}$

Any number, such as 2359, that is greater than 1000 but less than 10,000 is equal to 10 with an exponent whose value lies between 3 and 4 Any number, such as 892, that lies between 100 and 1000 is equal to 10 with exponent greater than 2 but less than 3 Any number between 10 and 100 is equal to 10 with an expo

nent between 1 and 2. Any number, such as 4, between 1 and 10 is equal to 10 with an exponent between 0 and 1. Any number, such as .6, between 0.1 and 1 *is equal to 10 with an exponent between* -1 and 0. Any number between .01 and .1 is equal to 10 with an exponent between -2 and -1. And any number between .001 and .01 is equal *to 10 with an exponent between* -3 *and* -2.

From this table you can see that only a small percentage of the positive numbers can be expressed as integral powers of 10. The remaining positive numbers are decimal powers of 10. For example, $18 = 10^{1.2553}$, $300 = 10^{2.4771}$, and $84.5 = 10^{1.9269}$.

The Characteristic and Mantissa[A]

The integral part of the logarithm is called the characteristic and the fractional part the mantissa. In the expressions $82 = 10^{1.9138}$ and $\log 82 = 1.9138$ the characteristic is 1 and the mantissa is .9138. In the expressions $467 = 10^{2.6693}$ and $\log 467 = 2.6693$ the characteristic is 2 and the mantissa is .6693.

Either the characteristic or the mantissa may be zero. In the expression $\log 100 = 2$ the mantissa is zero and in the expression $\log 6 = .7782$ the characteristic is zero. What number has the logarithm whose characteristic is zero and mantissa is zero?

How to Find Logarithms of Numbers between 1 and 10[A]

Since the logarithm of any number between 1 and 10 is greater than zero and less than 1, the characteristic of such a number is zero. *Table II at the back of the book gives the logarithms of numbers* from 1.00 to 9.99 inclusive to four decimal places. Can you tell why it is called a table of mantissas?

In the column headed N are the first two digits of the number. The third digit of the number is in the row with N. The logarithms of the numbers are below row N and to the right of column N. Now study the two examples.

Example 1. Find log 3.56.

Solution. In this example we are to find the power of 10 which equals 3.56. We know that the characteristic is zero. To find the logarithm of 3.56 we look in the column headed N (numbers) in Table II to find 3.5, which consists of the first two digits of 3.56. Then we follow the line horizontally to the right until we reach the column headed *6, the third digit of*

3 56 Here we find the number 5514 The decimal points which precede the logarithms are not written in the table We must supply them Placing the decimal point before 5514 we get 5514

Then $10^{0\ 5514} = 3\ 56$ or $\log 3\ 56 = 0\ 5514$

Example 2 Find log 7 8

Solution We write each number so that it has 3 digits In this case we add a zero to 7 8 to make it 7 80 We look in the column headed N to find the number 7 8 which consists of the first two digits of 7 80 Then we follow the line 7 8 horizontally to the right till we reach the column headed zero the third digit of 7 80 Here we find the number 8921 Writing the decimal point before this number we have 8921 which is log 7 8 Then $10^{0\ 8921} = 7\ 8$ or $\log 7\ 8 = 0\ 8921$

EXERCISES [A]

Express each of the following numbers as powers of 10

1 8 32	3 2 64	5 3 10	7 9 1
2 5 17	4 7 18	6 8 20	8 7 7

Find the logarithms of

9 2 9	11 8	13 4 99	15 7 13
10 1 4	12 6	14 9 85	16 4 11

How to Find the Antilogarithm (characteristic 0) [A]

The antilogarithm (abbreviated *antilog*) is the name of the number whose logarithm is given

Examples Since $\log 6\ 5 = 0\ 8129$ then antilog $8129 = 6\ 5$
Since $\log 8 = \ 9031$ then antilog $9031 = 8$

Finding the antilogarithm is finding the number whose logarithm is given This is done by reversing the steps in finding the logarithm when the number is given Now use pencil and paper when you study the next two examples

Example 1 Find the antilog of 8549

Solution In this example we are to find the number whose logarithm is 8549 or find the number that equals $10^{0\ 8549}$ We look among the logarithms in Table II to find the number

8549 We remember that the decimal points which precede the logarithms are omitted in the table We find 8549 in row 7.1 and in column headed 6 Then the antilog of 8549, or $10^{0.8549}$, is 7.16

Example 2 Find the antilog of 7125

Solution We look among the logarithms in Table II to find 7125 This number is not in the table but the number in the table nearest it is 7126 The number 7126 is the logarithm of 5.16 The antilog 7126 = 5.16, to the nearest hundredth

[A] **EXERCISES**

Find the numbers to the nearest hundredths which equal the following

1. $10^{0.8209}$	3. $10^{0.5563}$	5. $10^{0.8306}$	7. $10^{0.9400}$
2. $10^{0.9694}$	4. $10^{0.3366}$	6. $10^{0.8135}$	8. $10^{0.9996}$

Find the antilogarithms of the following logarithms to the nearest hundredths

9. 0.9375	11. 0.9475	13. 0.0253	15. 0.8670
10. 0.7778	12. 0.1139	14. 0.3263	16. 0.8888

How the Logarithm of a Number Changes When the Number Is Multiplied by a Power of 10 [A]

You have learned how to find the logarithms of numbers ranging from 1.00 to 10.00 and to find these numbers when their logarithms are known We shall now prove a theorem which will aid us in finding the logarithms of numbers less than 1 and the logarithms of numbers greater than 10 It will also aid us in finding these numbers when their logarithms are given

Theorem. If a positive number N is multiplied by the nth power of 10, its logarithm is increased by n

PROOF Let $\log_{10} N = x$

Then $N = 10^x$

M_{10^n} $N \cdot 10^n = 10^{x+n}$

Or $\log (N \cdot 10^n) = x + n$

Substituting $\log_{10} N$ for x,

$$\log (N \cdot 10^n) = \log_{10} N + n$$

If the number is multiplied by a negative nth power of 10 (divided by 10^n), the logarithm is decreased by n

Example 1 Given log 6 41 = 8069 Find log 641

Solution $641 = 6\ 41 \times 10^2$
By the theorem, log 641 = log 6 41 + 2
Then log 641 = 8069 + 2 = 2 8069

Example 2 If log 7 28 = 8621, find log 72 8

Solution. $72.8 = 7\ 28 \times 10^1$
By the theorem, log 72 8 = log 7 28 + 1
Then log 72 8 = 8621 + 1 = 1 8621

Example 3 If log 3 47 = 5403, find log 00347

Solution. $00347 = 3\ 47 \times 10^{-3}$
By the theorem, log 00347 = log 3 47 − 3
Then log 00347 = 5403 − 3

ORAL EXERCISES [A]

1. Given log 7 55 = 0 8779
   a Find log 755
   b. Find log 75 5
   c. Find log 7550
   d. Find log 755
2. Given log 1 72 = 2355
   a. Find log 17 2
   b Find log 172
   c. Find log 172
   d. Find log 1720
3. Given log 6 35 = 8028
   a Find log 6350
   b Find log 63 5
   c. Find log 0635
   d. Find log 635

How to Find the Logarithm of Any Three-Digit Number [A]

You have learned how to find the logarithm of a three-digit number whose value is greater than 1 and less than 10 We shall now learn how to find the logarithms of numbers like 523, 84, and 2750, whose values are less than 1 or greater than 10

If you have forgotten how to write numbers in standard notation, you should review pages 211–212

Now use pencil and paper as you study the three examples which follow

Example 1 Find log 847

Solution 1

	$847 = 8.47 \times 10^2$
From Table II	$8.47 = 10^{.9279}$
Then	$847 = 10^{.9279} \times 10^2$ $\quad 847 = 10^{2.9279}$
Then	$\log 847 = 2.9279$

Solution 2

	$847 = 8.47 \times 10^2$
By theorem p 293	$\log 847 = \log 8.47 + 2$
From Table II	$\log 8.47 = .9279$
Then	$\log 847 = 2.9279$

Example 2 Find log 92.8

Solution 1

	$92.8 = 9.28 \times 10$
From Table II	$9.28 = 10^{.9675}$
Then	$92.8 = 10^{.9675} \times 10$ $\quad 92.8 = 10^{\ .967}$
Then	$\log 92.8 = 1.9675$

Solution 2

	$92.8 = 9.28 \times 10^1$
By the theorem p 293	$\log 92.8 = \log 9.28 + 1$
From Table II	$\log 9.28 = .9675$
Then	$\log 92.8 = 1.9675$

Example 3 Find log 1760

Solution 1

	$1760 = 1.76 \times 10^3$
From Table II	$1.76 = 10^{.2455}$
Then	$1760 = 10^{.2455} \times 10^3$
	$1760 = 10^{3.2455}$
Then	$\log 1760 = 3.2455$

Solution 2

	$1760 = 1.76 \times 10^3$
By the theorem on page 293	$\log 1760 = \log 1.76 + 3$
From Table II	$\log 1.76 = .2455$
Then	$\log 1760 = 3.2455$

EXERCISES

(A)

1 When a number is written in standard notation how can you find the characteristic of its logarithm? How can you find its mantissa?

Find the logarithms of

2 436	5 85.6	8 7630	11 6.32
3 785	6 419	9 8100	12 504
4 92.7	7 84	10 93000	13 819

We shall now learn a shorter method of finding logarithms of numbers

Example 4 Find log 783

Solution We place a caret to the right of the first significant figure of the number, thus $7_\wedge 83$ Starting at the caret we count the number of decimal places to the decimal point We count 2 places to the right From Table II, log 7 83 = 8938 Then log 783 is 2 more than log 7 83 Then log 783 = 2 8938

Rule for Finding the Logarithm of a Number (the Power of 10 That Equals the Number)

1 Place a caret
 to the right of the first significant figure of the number
2 To find the characteristic (the integral part of the logarithm),
 start at the caret
 and count the number of places to the decimal point
 If you count to the right, it is positive;
 and if you count to the left, it is negative
3 Find the mantissa (the decimal power of 10) in Table II
4 Find the logarithm of the number
 by adding the characteristic and the mantissa

With practice you will be able to do Steps 1 and 2 mentally

Example 5 Find log 356

Solution Placing the caret to the right of the first significant figure, we have $3_\wedge 56$ Starting at the caret, we count two places to the right to the decimal point Then the characteristic is + 2 Considering the caret as a decimal point, we find in Table II that the logarithm of $3_\wedge 56$ is 5514 Adding the characteristic 2 to the mantissa 5514, we get 2 5514

Then $356 = 10^{2\ 5514}$, and the logarithm of 356 to base 10 is 2 5514

Example 6 Find log 58

Solution We place a caret to the right of the first significant figure, thus $5_\wedge 8$ Starting at the caret, we count one place to the right to the decimal point The characteristic is + 1

Considering the caret as a decimal point, we have the number 5 8 We add a zero to the number so that it will have three

digits, and get 5 80 To find the mantissa of the logarithm of 58, we find the logarithm of 5 80 in Table II We find that log 5 80 (which is the mantissa of the logarithm of 58) is 7634 We add the characteristic and mantissa and get 1 7634 Then $58 = 10^{1\ 7634}$ and log 58 = 1 7634

EXERCISES* [A]

Find the logarithms of

1 726	4 24 9	7 9 2	10 632	13 92 1
2 843	5 1320	8 346	11 21 8	14 540
3 860	6 4 75	9 3260	12 2180	15 700

Negative Characteristics [A]

You have learned how to find logarithms with positive characteristics and now we shall consider logarithms with negative characteristics Suppose that we wish to find log 0 58 We place the caret thus, $0\ 5_{\wedge}8$ Starting at the caret, we count to the left one place to the decimal point Then $0\ 58 = 5\ 8 \times 10^{-1}$, that is, 5 8 must be multiplied by 10^{-1} to equal 0 58 Then the logarithm of 58 is one less than the logarithm of 5 8 Hence the characteristic is -1

From the table, log 5 8 = 7634 We cannot write the logarithm of 0 58 in the form $-1\ 7634$ because the mantissa 7634 is positive Then how shall we write the logarithm? Shall we write it

in the form $-1 + 7634$?
or in the form $\bar{1}\ 7634$?
or in the form $-\ 2366$, which equals $-1 + 7634$?
or in the form $9\ 7634 - 10$?

We shall use the last form because it simplifies computation For example, if the characteristic is -3 and the mantissa is 2716, we write the characteristic in the form $7 - 10$ and the logarithm in the form $7\ 2716 - 10$, and if the characteristic is -4 and the mantissa is 4603, we write the logarithm in the form $6\ 4603 - 10$

EXERCISES [A]

Write the following numbers with their exponents in the -10 form

1 10^{8732-1}	3 10^{3614-3}	5 $10^{-3+4784}$
2 $10^{-2+6481}$	4 $10^{-4+8891}$	6 $10^{-1+1215}$

Write the following logarithms in the − 10 form

7. 3416 − 1	9. − 1 + .2316	11. 5278 − 6
8. $\bar{2}$ 6146	10. − 4 + 6325	12. 4913 − 4

Example Find log 0 023

Solution. Placing the caret to the right of the first significant figure, we have 02‸3 Starting at the caret, we count two places to the left to the decimal point Then the characteristic is − 2, or 8 − 10 In Table II we find that log 2 3 = 3617 Then log 0 023 = 8 3617 − 10

[A]

EXERCISES

Find the logarithms of

1. 0 563	5. 0 719	9. 0 125	13. 0 00728
2. 0 981	6 0 514	10. 0 063	14. 0 00443
3. 0 068	7. 0 472	11. 0 047	15. 0 728
4. 0 0143	8. 0 971	12 0 561	16. 0 003

Finding the Antilogarithm When the Characteristic Is Not Zero[A]

On page 292 you learned how to find the number when its logarithm is known and has a zero characteristic The following rule can be used to find the antilogarithm in all cases

To Find a Number When Its Logarithm Is Given

1 Use Table II to find the number whose logarithm is the mantissa of the given logarithm Use a caret (to the right of the first significant figure of the number) as the decimal point

2. Starting at the caret, count (to the right if the characteristic is positive and to the left if it is negative) the number of places equal to the absolute value of the characteristic to find the position of the decimal point of the number

Example 1 Find the number whose logarithm is 2 8370

Solution We look in Table II to find the mantissa 8370 It is in row 6‸8 and in column 7 Then 8370 is the logarithm of 6‸87 We use the caret as a temporary decimal point From the caret we count two places to the right to find the position of the decimal point and get 687

Then antilog 2 8370 = 687

Example 2 Find the antilog of 7.7060 − 10

Solution We search among the logarithms of Table II for the number 7060. There is no such number in the table. The nearest number to it is 7059. The number 7059 is the logarithm of 5.08. We place the caret thus, $5_{\wedge}08$.

Since the characteristic is 7 − 10, or −3, we count to the *left* three places from the caret to find the final position of the decimal point. We get 0.00508 as the result.

Then antilog 7.7060 − 10 = 0.00508 to the nearest hundred thousandth.

[A]

EXERCISES

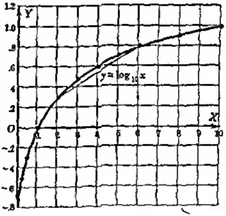
Find the antilogarithms of

1. 2.8476	4. 3.6571	7. 9.6514 − 10	10. 3.3729
2. 1.8021	5. 1.7101	8. 9.7076 − 10	11. 7.8615 − 10
3. 2.5635	6. 0.4045	9. 7.1645 − 10	12. 8.9304 − 10

The Logarithmic Curve [A]

The logarithmic curve shown at the right will help you to remember many facts you have already learned.

The graph does not give a true picture of the relationship between a number and its logarithm because the x- and y-units are different. You may like to construct the graph with the same scale on each axis.



[A]

ORAL EXERCISES

1. What is the value of log 1? of log 10?

2. When a number becomes larger, how does its logarithm change?

3. Complete: The logarithm of a number greater than __?__ is positive, and the logarithm of a number between __?__ and __?__ is negative.

Facts You Should Know about Logarithms

I The mantissas of two numbers are the same when the numbers differ only in the positions of their decimal points

II The characteristic depends upon the position of the decimal point For each place the decimal point of a number is moved to the right, the characteristic (and logarithm) is increased by 1, and for each place the decimal point is moved to the left, the characteristic (and logarithm) is decreased by 1

III. The characteristic of a number between 1 and 10 is zero

IV. To find the logarithm of a number

1. Place a caret to the right of the first significant figure

2. To find the characteristic, start at the caret and count the number of places to the decimal point of the number If you count to the right, it is positive, and if you count to the left, it is negative

3 Considering the caret as a decimal point, find the mantissa in Table II

V. To find the antilogarithm

1. Use Table II to find the antilogarithm of the mantissa of the logarithm Use a caret as the temporary decimal point of the antilogarithm

2. Starting at the caret, count (to the right if the characteristic is positive, and to the left if it is negative) the number of places equal to the absolute value of the characteristic to find the position of the decimal point

How to Find the Logarithm by Interpolation

Table II at the end of the book gives the logarithms of numbers of three digits from 1 00 to 9 99 Since the mantissas of numbers which differ only in the position of the decimal point are the same, Table II gives the mantissas of numbers with three significant figures

The mantissas of numbers with more than three significant figures can be found approximately by the method known as <u>interpolation</u>

Suppose that we wish to find the logarithm of 48 361. We round the number off to 48 36 so that it has four significant figures The

characteristic is 1 From Table II we can find the logarithm of 4 836 Let us forget the decimal point of 4 836 while interpolating and write it 4836 From the table the mantissa of 483 is 6839 and the mantissa of 484 is 6848 Let us add a zero to 483 and to 484, making them 4830 and 4840 Let us arrange the integers 4830 to 4840 in a table

4840	6848	
4839		
4838		
4837		
4836	x	
4835		9
4834		
4833		
4832		
4831		
4830	6839	

From this table 4836 is $\frac{6}{10}$ of the way from 4830 to 4840 Then x, the mantissa of 4836, will be approximately $\frac{6}{10}$ of the way from 6839 to 6848 The <u>tabular difference</u> = 6848 − 6839 = 0009 In writing the tabular difference, we omit the decimal point and zeros Thus a tabular difference of 8 means 0008 Then $x = 6839 + \frac{6}{10}(0009)$ $= 6839 + 0005 = 6844$ Notice that $\frac{6}{10}$ of 0009 is 00054, and that the result is rounded off to four decimal places, making it 0005

Then log 48 361 = 1 6844

In using a four place table, such as Table II, all numbers should be rounded off to four significant figures When rounding off a number ending in 5 or in 5 followed by zeros, either the 5 is dropped or 1 is added to the preceding digit so as to make it *even*

To Find the Mantissa by Interpolation

1 Round off the number to four significant figures.
2. Find the mantissa of the first three digits.
3 Multiply $\frac{1}{10}$ of the tabular difference by the fourth digit
4 Add the results of steps 2 and 3

Example 1 Find log 37 654

Solution 37 654 rounded off to four significant figures becomes 37 65 The characteristic is $+1$

$$\begin{array}{ll} 3770 & 5763 \\ 3765 & x \quad 11 \\ 3760 & 5752 \end{array}$$

$$x = 5752 + \tfrac{5}{10}(\ 0011) = 5752 + \ 0006$$
$$x = 5758$$
$$\log 37\ 65 = 1\ 5758$$

Example 2 Find log 488 5

Solution The characteristic is 2

$$\begin{array}{ll} 4890 & 6893 \\ 4885 & x \quad 9 \\ 4880 & 6884 \end{array}$$

$$x = 6884 + \tfrac{5}{10}(\ 0009) = 6884 + \ 0004$$
$$x = 6888$$
$$\log 488\ 5 = 2\ 6888$$

Example 3 Find log 8 047

Solution The characteristic is 0

$$\begin{array}{ll} 8050 & 9058 \\ 8047 & x \quad 5 \\ 8040 & 9053 \end{array}$$

$$x = 9053 + \tfrac{7}{10}(\ 0005) = 9053 + \ 0003$$
$$x = 9056$$
$$\log 8\ 047 = 0\ 9056$$

NOTE Since $\frac{7}{10}(\ 0005)$ is 0 00035 you would think that 0004 would be added to 9053 in Example 3 We do not round off the correction (00035) so that *it* is an even number instead we round off the correction so that the *mantissa is an even number*

EXERCISES

[A]

Find the logarithms of

1 41 23	5 3 141	9 0 0235	13 797 7
2 5 265	6 0 4965	10 0 91535	14 0 82476
3 321 5	7 5 0821	11 1004	15 0 04025
4 7 734	8 7 1936	12 0 9417	16 0 071265

How to Find the Antilogarithm by Interpolation[A]

By reversing the process of finding the logarithm of a number, we can find the antilogarithm when the logarithm is known.

Example Find the antilogarithm of 1.5445.

Solution Table II does not contain the mantissa .5445. The mantissa .5445 lies between the mantissas .5441 and .5453. We find that the antilog of .5453 is 3.51 and the antilog of .5441 is 3.50. We ignore the decimal points of the antilogs and add a zero to each one. Then we make the table

3510	5453	
x	5445	12
3500	5441	

The tabular difference is 12. Then 5445 is $\frac{4}{12}$, or $\frac{1}{3}$, of the way from 5441 to 5453. Then x is approximately $\frac{1}{3}$ of the way from 3500 to 3510. Then $x = 3500 + \frac{1}{3}(10) = 3503$. Now we place the caret to the right of the first significant figure thus: $3_\wedge503$. Since the characteristic is 1, the antilog of $1.5445 = 35.03$.

EXERCISES

Find the numbers whose logarithms are

1. 3.0531	5. 0.8671	9. 7.7056 − 10	13. 3.4261
2. 1.7782	6. 1.6170	10. 2.1846	14. 0.4718
3. 2.8094	7. 0.4130	11. 9.8546 − 10	15. 9.5216 − 10
4. 0.8745	8. 3.1732	12. 8.2413 − 10	16. 8.8454 − 10

Laws of Logarithms[A]

The laws of logarithms are derived from the laws of exponents. We shall state four of these laws without proof.

1. The logarithm of a product equals the sum of the logarithms of its factors.

$$\log (MN) = \log M + \log N$$

2. The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.

$$\log \left(\frac{M}{N}\right) = \log M - \log N$$

3. The logarithm of the nth power of a number equals n times the logarithm of the number.

$$\log M^n = n \log M$$

4 The logarithm of the real positive *n*th root of a number equals $\frac{1}{n}$ times the logarithm of the number

$$\log \sqrt[n]{M} = \frac{1}{n} \log M.$$

Computing with Approximate Numbers [A]

When approximate numbers are added, subtracted, multiplied, or divided, the results are approximate numbers Since most logarithms are approximate numbers, the results obtained by computing with logarithms are approximate Various rules may be used for computing with approximate numbers In this text we shall follow rules which are sufficiently accurate and easily applied They follow

Rules for Adding (or Subtracting) Approximate Numbers

1 Round them off to the same unit of measurement and add (or subtract)
2 The sum (or difference) will have the same unit of measurement as the numbers added (or subtracted)

Example Add the approximate numbers 13 6, 14 32, and 19 65

Solution The quantities should be measured by the same unit of measure The smallest unit of measure common to all three numbers is 1 Then we round off 14 32 and 19 65 to tenths Adding 13 6, 14 3 and 19 6, we get 47 5 as the sum

13 6	
14 3	
19 6	
47 5	the sum

The product (or quotient) of two approximate numbers should not contain more significant figures than the given number containing the fewer significant figures

Example Multiply 2 461 by 1 4

Solution We multiply the numbers and round off the product so that it has the same number of significant figures as 1 4 The same result can be had by rounding off 2 461 to 2 46 and multiplying 2 46 by 1 4

2 461
1 4
9844
2461
3 4454, or 3 4

Computing with Logarithms [A]

As you compute with logarithms you should remember that logarithms are exponents and that they are also approximate numbers.

Example 1 Multiply 47.2 by 3.61

Solution Let $x = 47.2 \times 3.61$

Then $\log x = \log 47.2 + \log 3.61$

$$\begin{aligned} \log 47.2 &= 1.6739 \\ \underline{\log 3.61} &\underline{= 0.5575} \\ \log x &= 2.2314 \\ x &= 170.4 \end{aligned}$$

If 47.2 and 3.61 are approximate numbers, 170.4 should be rounded off to 3 significant figures. Then $x = 170$.

Example 2 Multiply 79.43 by .057

Solution Let $x = 79.43 \times .057$

Then $\log x = \log 79.43 + \log .057$

$$\begin{aligned} \log 79.43 &= 1.9000 \\ \underline{\log .057} &\underline{= 8.7559 - 10} \\ \log x &= 0.6559 \\ x &= 4.528 \end{aligned}$$

Example 3. Divide 48,200 by 0.913

Solution Let $x = 48{,}200 \div 0.913$

Then $\log x = \log 48200 - \log 0.913$

$$\begin{aligned} \log 48200 &= 14.6830 - 10 \\ \underline{\log .913} &\underline{= 9.9605 - 10} \\ \log x &= 4.7225 \\ x &= 52790 \end{aligned}$$

The characteristic of log 48200 is 4. It is written in the form $14 - 10$ to make subtraction easier.

Example 4 Divide -4.563 by 13.22

Solution When logarithms are used to compute with negative numbers, we calculate as if the numbers are positive. Then we give the proper sign to the result.

Let $\qquad x = 4.563 \div 13.22$ (cont on p 306)

Then $\log x = \log 4\ 563 - \log 13\ 22$

$$
\begin{aligned}
\log 4\ 563 &= 10\ 6593 - 10 \\
\underline{\log 13\ 22} &= \underline{\ \ 1\ 1213\qquad} \\
\log x &= \ \ 9\ 5380 - 10 \\
x &= \ \ 0\ 3452 \\
-4\ 563 \div 13\ 22 &= -0\ 3452
\end{aligned}
$$

Why was the characteristic of the logarithm of 4 563 written in the form $10 - 10$?

Example 5 Find the value of $7\ 1632^3$

Solution We round off 7 1632 so that it has four significant figures

Let $x = 7\ 163^3$

Then

$$
\begin{aligned}
\log x &= 3 \log 7\ 163 \\
\log 7\ 163 &= 0\ 8551 \\
\log x &= 2\ 5653 \\
x &= 367\ 5
\end{aligned}
$$

Example 6 Find the cube root of 0682

Solution Let $x = \sqrt[3]{0682}$

$$
\begin{aligned}
\log\ 0682 &= 28\ 8338 - 30 \\
\log x &= 9\ 6113 - 10 \\
x &= 0\ 4086
\end{aligned}
$$

The characteristic of log 0682 is written $28 - 30$ so that the negative part will be exactly divisible by 3

In the following exercises the numbers are exact

EXERCISES

[A]

Compute by logarithms giving the answers to the nearest third significant figure

1 $3\ 44 \times 272$	8 91×172	15 $(19\ 4)(3\ 46)$
2 $9\ 21 \times 326$	9 200×612	16 $\sqrt{0673}$
3 $72\ 8 \div 3\ 62$	10 $\sqrt[3]{94}$	17 $\sqrt[6]{343}$
4 $603 \div 373$	11 $4\ 58 \div 76$	18 $\sqrt[7]{728}$
5 $\sqrt{870}$	12 $100 \div 53$	19 $90\ 8 \div 74$
6 $305 \div 2\ 34$	13 $4 \times 3\ 16$	20 $63\ 4 \div 1\ 72$
7 84×92	14 $27 \times 0\ 093$	21 $283 \times 3\ 46$

22. 365×941
23. $27.5 + 382$
24. 76×403.5
25. $45.82 + 50.67$
26. -40.7×5029
27. $-603.4 + 3.75$
28. 315.4^2
29. $\sqrt[3]{425.5}$
30. $5.83 \times 7.6 \times .07$
31. $4.3 \times 1.07 \times 9801$
32. $(-7.063)^3$
33. $\sqrt[6]{785.1}$
34. $1000 \div 3.14$
35. $(3.1416)^3$
36. 7.842×3.14

Example 7 Find the value of $\sqrt{\dfrac{47.2 \times 9.6}{0.032}}$

Solution

$$
\begin{aligned}
\log 47.2 &= 1.6739 \\
\log 9.6 &= 0.9823 \\
\hline
\log \text{product} &= 12.6562 - 10 \\
\log 0.032 &= 8.5051 - 10 \\
\hline
\log \text{fraction} &= 4.1511 \\
\log \sqrt{\text{fraction}} &= 2.0756 \\
\text{antilog } 2.0756 &= 119.0
\end{aligned}
$$

Then $$\sqrt{\frac{47.2 \times 9.6}{0.032}} = 119.0$$

The zero is written to the right of the decimal point to show that the answer is correct to four significant figures

[B]

Compute with logarithms

37. $\sqrt{\dfrac{27.35}{.064 \times 7.89}}$
38. 9.96×3.15^2
39. $(-0.0578)^3$
40. $\dfrac{5.317 \times 4.787}{18}$
41. $\dfrac{7.63 + 6.28}{.017}$ colog colog
42. $\sqrt[5]{\dfrac{762.3}{14.9 \times 0.47}}$
43. $\sqrt[3]{84.9 \times 1.04^2}$
44. $\sqrt[4]{\dfrac{2.316 \times 1.7}{9.64}}$
45. $(4832 \times 1.67)^{\frac{1}{3}}$
46. $(1.732 \times 1.414)^{\frac{2}{3}}$
47. $\dfrac{1.006 \times 5000}{328.6}$
48. $\sqrt{1.04^3 \times 75^2}$ 3 · log 1.04 + 2 l

USING LOGARITHMS

[A]

1 If $\log 5 = 0\ 6990$, find without using tables $\log 25$, $\log 125$, $\log 500$, $\log\ 05$ 3960

2 $\log 6 = 0\ 7782$ and $\log 8 = 0\ 9031$ Find without the use of tables $\log 36$, $\log 216$, $\log 64$, $\log 48$, $\log \frac{4}{3}$, $\log\ 75$, $\log 384$

3 Find the area of a triangle whose base is 16 4 inches and whose altitude is 8 9 inches

4 Find the altitude of a triangle if its area is 380 square inches and its base is 47 inches

5 Find the area of an equilateral triangle if a side is 11 6 inches $\left(A = \frac{s^2}{4}\sqrt{3}\right)$

6 Find the side of an equilateral triangle whose area is 125 square inches $\left(\text{Solve } A = \frac{s^2}{4}\sqrt{3} \text{ for } s \text{ and use logarithms}\right)$

7 How many cubic yards of earth must be removed for a cellar 30 feet long, 26 feet wide, and 7 feet deep?

8 Find the number of gallons of water needed to fill a pool 30 feet wide and 60 feet long to an average depth of 5 feet (1 cu ft = 7 48 gal)

9 Using the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere, find r when $V = 50$ cubic inches and $\pi = 3\ 1416$

10 $S = \frac{1}{2}gt^2$ Find t when $S = 400$ Use $g = 32\ 16$

11 $v = \sqrt{2gs}$ is a formula for uniformly accelerated motion Find v when $s = 65$, using 32 for g

12 Chapter 12 of this text is a study of progressions One formula in this chapter is $l = ar^{n-1}$ Find l when $a = 9$, $r = 2$, and $n = 12$

13 The formula $t = \pi\sqrt{\frac{l}{g}}$ gives the time, in seconds, of the swing of a pendulum when its length l, in feet, is known Find the time between the ticks of a clock if the length of its pendulum is 18 inches (Use $g = 32$)

14 $S = \frac{\pi r^2 E}{180}$ is a formula for finding the area of a triangle on a sphere Find S when $r = 3\ 44$ and $E = 75$, using 3 14 for π

15 Find the area of a triangle having sides 18 26 and 31 Use the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ the sides being a b and c, and $s = \frac{1}{2}(a+b+c)$

Compound Interest [A]

If money is loaned for a long period of time the interest when due is usually added to the principal to form a new principal Loan companies usually compound the interest semiannually

If p dollars is invested at 4% compounded semiannually then 2% of p is added to p at the end of the first 6 months Then the amount at the end of 6 months is $1\ 02\,p$ This sum becomes the principal for the next 6 months At the end of the year the amount is $1\ 02(1\ 02\,p)$ Continuing this reasoning we find that the amount of p dollars for n periods at r% per period is given by the formula $A = p(1+r)^n$

Example Find the amount of $375 for 8 years at 4% compounded semiannually

Solution In this problem $r = 2\%$ and $n = 16$

$$A = 375(1 + \ 02)^{16} = 375(1\ 02)^{16}$$

$$\begin{aligned} \log 1\ 02 &= 0\ 0086 \\ \log 1\ 02^{16} &= 0\ 1376 \\ \underline{\log 375} &\underline{= 2\ 5740} \\ \log A &= 2\ 7116 \\ A &= 514\ 8 \end{aligned}$$

The amount is $514 80

(B) **PROBLEMS**

Find the amount using logarithms

1 $400 compounded annually for 6 years at 4% per year

2 $875 compounded annually for 8 years at 6% per annum

3 $2000 compounded semiannually for 10 years at 4% per annum

4 $1500 compounded semiannually for 14 years at 3% per year

5 $6500 compounded semiannually for 7 years at 3% per year

6 $840 compounded semiannually for 9 years at 5% per year

7 \$500 compounded quarterly for 6 years at 4% per year

8 \$650 compounded quarterly for 9 years at 6% per year

9 Manhattan Island was purchased about 330 years ago for \$24 If this money had been invested at 3% per year, compounded annually, what would its present value be?

10 How many dollars must be invested at 4%, compounded annually, to amount to \$1000 in 20 years? (We have $1000 = p[1 + .04]^{20}$ Then $\log 1000 = \log p + 20 \log 1.04$)

11 How much money must be invested for 15 years at 4%, compounded semiannually, to amount to \$2500?

12 The Merchants Building and Loan Association pays 3% and compounds the interest every 6 months How much money must be invested in it to amount to \$4000 in 10 years?

Exponential Equations [8]

An exponential equation is one in which the unknown occurs in the exponent Thus $2^x = 16$ and $5^x = 125$ are exponential equations If x is a rational number, it can often be found by inspection If x is an irrational number, we use logarithms to solve the equation

Example 1 Solve $5^x = 125$

Solution We write 125 as a power of 5

Then $5^x = 5^3$

Then $x = 3$

Example 2 Solve $2^{-x} = 32$

Solution

$$2^{-x} = 32$$
$$2^{-x} = 2^5$$
$$-x = 5$$
$$x = -5$$

Example 3 Solve $4^x = 15$

Solution 15 cannot be written as a rational power of 4 We find the logarithms of both members obtaining

$$x \log 4 = \log 15$$
$$x = \frac{\log 15}{\log 4}$$
$$x = \frac{1.1761}{0.6021}$$

$$\begin{aligned} \log 1.1761 &= 10.0704 - 10 \\ \log 0.6021 &= 9.7797 - 10 \\ \hline \log x &= 0.2907 \\ x &= 1.953 \end{aligned}$$

You should note that $\dfrac{\log 15}{\log 4}$ is not the same as $\log \dfrac{15}{4}$ [8]

EXERCISES

Solve

1 $2^x = 16$ =4	6 $4^x = 16$	11 $\log 2x = 1.6435$
2 $2^x = 64$ =6	7 $5^{x-2} = 125$ 5	12 $\log x^2 = 2.2279$
3 $3^x = 21$ 2.771	8 $6^{-x} = 216$	13 $3^{2x+1} = 245$
4 $3^{-x} = 243$	9 $7^x = 343$	14 $1.24^x = 2.3$
5 $3^{x-2} = 81$	10 $2^{x-6} = 8$ 9	15 $1.46^x = 1.96$

Change of Base [8]

It is sometimes necessary to find the logarithm of a number to a different base For example, suppose we wish to find the logarithm of 50 to the base 2

Let $\qquad x = \log_2 50$

Then $\qquad 2^x = 50$

Taking logarithms of both members,

$$x \log_{10} 2 = \log_{10} 50$$

$$x = \frac{\log_{10} 50}{\log_{10} 2} = \frac{1.6990}{0.3010} = 5.644$$

500
301

This is a special case of the following theorem

Theorem The logarithm of a number, N, to the base b is equal to the logarithm of the number to the base a divided by the logarithm of b to the base a

Let $\qquad x = \log_b N$

Then $\qquad b^x = N$

$$x \log_a b = \log_a N$$

$$x = \frac{\log_a N}{\log_a b}, \quad \text{or} \quad \log_b N = \frac{\log_a N}{\log_a b}$$

Sometimes we state the theorem in the form $x = \log_a N \cdot \log_b a$

EXERCISES

[B]

Find the logarithms of the following numbers

1 10 to the base 2
2 20 to the base 3
3 75 to the base 4
4 120 to the base 20
5 10 to the base 4
6 1300 to the base 8
7 750 to the base 5
8 10 to the base 5
9 Show that $\log_{10} 75 = \dfrac{1}{\log_{75} 10}$

MISCELLANEOUS EXERCISES

[A]

1 If log 7 55 = 0 8779, find log 755
2 If log 6 = 7782, find log 6000
3 If log 172 = 2 2355, find log 1 72
4 If log 7 29 = 8627 find log 0729
5 If log 6 35 = 8028, find log 0635
6 If log 6 = 7782 and log 2 = 3010, find log 12
7 If log 20 = 1 3010 and log 5 = 0 6990, find log 4
8 If log 9 = 0 9542, find log 81
9 If log 9 = 0 9542, find log 3
10 If log 80 = 1 9031, find log $\sqrt{80}$

[B]

The following exercises can be solved without the use of a table Can you solve them?

11 The log of 27 is 1 4314 and the log of 4 is 0 6021 Find log 108

12 The log of 7640 is one of the following $\frac{1}{7640}$, 4 067, 3 8831, or 7 640 Which is it?

13 The log of 2 is 0 3010 Find log $\sqrt[5]{2}$

14 log 9 = 0 9542 Find log 300

15 log 3 = 0 4771 and log 4 = 0 6021 Find log 128

16 The log of 2 is 0 3010 Find the log of 16

17 If log 30 = 1 4771, find log 0 03

18 Does $\log xy = \log x \;\; \log y$?

19 If log 6 72 = 0 8274 find log 67200

20 If $\log 210 = 2.3222$ and $\log .07 = 8.8451 - 10$, find $\log 3$

21 If $\log 2 = 0.3010$, find $\log 5$

22 If $\log 0.528$ is $9.7226 - 10$, what is the log of $\sqrt{52.80}$?

23 $\log 25 + \log 4$ equals one of the following 100, $\log 29$, 10, or 2 Which is it?

24 If $\log 700 = 2.8451$, find $\log (\frac{1}{7})$

25. Given $\log 12 = 1.0792$ and $\log 625 = 2.7959$, find $\log 6$

[C]

26 Find the value of $\log_{125} 4 - \log_2 125$

27 If $\log_{10} x^2 - 2x + 7 = 1$, find x

28 Find the value of $\log_9 \sqrt{27}$

29 Find the value of $\log_a 1 + \log_8 4 \sqrt[5]{16}$

30 $\log_{10} 5 = 0.6990$ How many digits are there in the result when 5 is raised to the 20th power?

THE SLIDE RULE

The slide rule is an instrument used to solve problems involving multiplication, division, squares, cubes, square roots, cube roots, or proportion It performs multiplications of numbers by mechanically adding their logarithms, and divisions of numbers by mechanically subtracting the logarithms of the divisors from the logarithms of the dividends It squares numbers by doubling their logarithms How do you think it can be used to find the square roots of numbers?

Keuffel & Esser Co.

Polyphase Slide Rule

The Parts of a Slide Rule[A]

A slide rule consists of three parts,—the fixed part or *rule* proper, the *slide*, and the *runner* The rule of the slide rule shown on the preceding page contains three logarithmic scales, A, D, and K The slide contains three logarithmic scales, B, C, and CI, on its face and two logarithmic scales, S and T, on its back

Scales C and D, which are identical, are used in multiplication and division, and scales A and D are used in squaring numbers and in finding their square roots The K scale is used with the D scale in cubing numbers and in finding their cube roots The CI scale gives the reciprocals of numbers on the C scale Scale S on the back of the slide is used to find the sines of angles, and scale T on the back of the slide is used to find the tangents of angles The simplest slide rule (the Mannheim slide rule) has only six logarithmic scales—A, B, C, D, S, and T It may have the scale L, which is the scale of equal parts

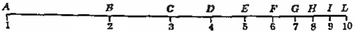
The runner is often made of glass and metal with a hairline on the surface of the glass The hairline is used to align the scales

The Logarithmic Scale[A]

Before we learn how to read a logarithmic scale, let us study the principle which is used in its construction Below is a table of logarithms (correct to two decimal places) of the integers 1 to 10 inclusive

Numbers	1	2	3	4	5	6	7	8	9	10
Logarithms	0	30	48	60	70	78	85	90	95	1

On a strip of paper draw any convenient line segment AL We chose the segment below, which is 10 centimeters long



Since $\log 1 = 0$ and $\log 10 = 1$, we label A with 1 and L with 10 Remember that AL is one unit long Next we plot the logarithms of the other numbers of the table For example, the logarithm of 2 is represented by AB, which is 30 of AL Then AB, which is 3 centimeters long, represents log 2 Likewise, AC represents log 3, AD represents log 4, etc

The scale formed on AL is called a logarithmic scale. Do you see that a logarithmic scale is similar to Table II in that it gives the logarithms of numbers from 1 to 10?

Suppose that we wish to find log 60 on our logarithmic scale. We know that $\log 60 = \log 10 + \log 6$. Now $\log 10 = AL$ and $\log 6 = AF$. Then $\log 60 = AL + AF$. Since $AL = 1$ and AF is a fraction less than 1, we know that AL is the characteristic of the logarithm of 60 and AF is the mantissa.

We use the logarithmic scale, just as we use Table II, to find the mantissas of numbers.

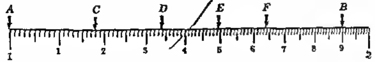
Then AF on the scale represents any number of the form 6×10^n, n being an integer.

To simplify operations on a slide rule the number 10 is replaced by the number 1. Then the scale starts with 1 and ends with 1. Each of the two 1's which are at the ends of the scale is called an index (plural *indices*). We denote the left index by 1 L and the right index by 1 R.

How to Read the Scales [8]

Let us learn how to read scale D, which is identical with scale C. After you have learned to read scale D, the reading of the other scales will not be difficult.

Using the figure on page 313, you can see that scale D is divided into 9 unequal parts, or units. Unit one extends from large 1 to large 2, unit two from large 2 to large 3, and so on. Let us see how each of these units is divided.

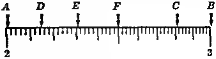


Unit one starts at large 1 and ends at large 2. It is divided into 10 unequal parts, the division points being numbered by the smaller digits 1, 2, 3, etc. Each of these parts represents 0.1 of a unit. Each of these parts (0.1 units) is divided into 10 parts, each of which represents 0.01 of a unit.

In the diagram above, the A reading is 1, the B reading is 19, the C reading is 118, the D reading is 134, the E reading is 1495, and the F reading is 164.

Unit two extends from large 2 to large 3 It is the same as unit one except that the subdivisions are not numbered and each of the smallest parts represents 0 02 of a unit instead of 0 01 of a unit

In unit two, illustrated at the right, the A reading is 2, the B reading is 3, the C reading is 28, the D reading is 214, the E reading is 23, and the F reading is 249 Unit three is like unit two except that the divisions are smaller



Each of the units four, five, six, seven, eight, and nine is divided into ten parts, each of which is divided into two parts Each of the smallest parts represents 0 05 of a unit

How to Multiply with the Slide Rule[A]

As you know, two numbers are multiplied by adding their logarithms Use your slide rule in studying Examples 1, 2, and 3

Example 1 Multiply 3 by 2

Solution Place 1 L (left index) of scale C above 3 of scale D Use the hairline of the runner to find the number of scale D that is below 2 of scale C The number is 6 Then $3 \times 2 = 6$

Example 2 $24 \times 18 = ?$

Solution Place 1 L of scale C above 24 of scale D Use the hairline to find the number on scale D that is below 18 on scale C It is 432 The digit 2 is estimated Since 18 is more than 10 and less than 20, the product is more than 240 and less than 480, that is the product has three digits to the left of the decimal point Then $24 \times 18 = 432$

Example 3 $74\ 5 \times 4\ 6 = ?$

Solution If we place 1 L of scale C over 745 of scale D, the 46 of scale C extends beyond scale D In such a case we place 1 R (right index) of scale C over 745 of scale D There, as before, we find the number on scale D that is below 46 on scale C The number is 343 By inspection, the product has three digits to the left of the decimal point Then $74\ 5 \times 4\ 6 = 343$ In general, results are only accurate to three significant figures

[A] **EXERCISES**

Multiply using the slide rule

1 8×2	4 65×15	7 49×52	10 455×65
2 5×4	5 18×48	8 63×41	11 780×47
3 6×30	6 75×70	9 77×84	12 843×125

How to Divide with the Slide Rule [A]

Division is performed by subtracting the logarithm of the divisor from the logarithm of the dividend Study the following solutions

Example 1 $60 \div 4 = ?$

Solution Place the 4 of scale C above 60 of scale D Then find the number on scale D that is below 1 L of scale C The number is 15

Example 2 Divide 85 by 20

Solution Place 20 of scale C above 85 of scale D Then find the number on scale D that is below 1 L of scale C It is 425 By inspection $85 \div 20 = 4\ 25$

Example 3 Divide 364 by 4 75

Solution When 4 75 of scale C is placed above 364 of scale D 1 L of scale C extends beyond scale D So we find the number on scale D that is opposite 1 R of scale C The number is 766 By inspection the quotient has two integral digits Then $364 \div 4\ 75 = 76\ 6$

[A] **EXERCISES**

Divide using the slide rule

1 $84 \div 6$	4 $625 \div 25$	7 $125 \div 50$	10 $186 \div 53$
2 $42 \div 3$	5 $750 \div 30$	8 $46 \div 115$	11 $22\ 5 \div 63$
3 $81 \div 27$	6 $76 \div 19$	9 $256 \div 32$	12 $191 \div 65$

How to Square a Number with the Slide Rule [A]

Scales A and D are used to square a number

Example 1 Find the value of 75^2

Solution Find 75 on scale D Opposite it on scale A read 5620 By inspection the square of 75 has 4 digits Then $75^2 = 5620$ The actual value of 75^2 is 5625

Example 2 Square 26 1

Solution Find 261 on scale D Opposite it on scale A read 681 By inspection the square has 3 integral digits Then $26\ 1^2 = 681$

How to Find the Square Root of a Number with the Slide Rule [A]

Two examples and their solutions will be given

Example 1 Find the value of $\sqrt{784}$

Solution First point off the number as you did in finding the square root arithmetically, thus $\overline{7}\ \overline{84}$ The first group, 7, has one digit When the first group has one digit, we use the left half of scale A We find 784 on the left half of scale A Opposite it on scale D we find 28 That is $\sqrt{784} = 28$

Example 2 $\sqrt{56\ 7} = ?$

Solution We point off the number thus $\overline{56}\ \overline{70}$ The first group has 2 digits therefore we use the right half of scale A We find 567 on the right half of scale A Opposite it on scale D is 753 We know that the root has one integral digit Then $\sqrt{56\ 7} = 7\ 53$

[A]

Square the following

1 8	**4** 25	**7** 95	**10** 86 3
2 6	**5** 68	**8** 632	**11** 2 59
3 7	**6** 82	**9** 471	**12** 64 8

Find the positive square roots of the following

13 625	**16** 2025	**19** 922
14 4225	**17** 2 89	**20** 220
15 7225	**18** 205	**21** 96 04

How to Solve a Proportion with the Slide Rule [A]

The solution of any simple proportion requires no operation other than multiplication and division

For example, the proportion $\frac{x}{4} = \frac{2}{6}$, in which $x = \frac{4 \times 2}{6}$, can be solved in these ways by multiplying 4 by 2 and dividing the product by 6 or by dividing 4 by 6 and multiplying the quotient by 2, or

by dividing 2 by 6 and multiplying the quotient by 4, or by dividing 6 by 2 and then dividing 4 by the quotient

When using the slide rule the last procedure is the shortest and most easily remembered See how easy it is We leave the proportion $\frac{x}{4}=\frac{2}{6}$ as it is written We use scale C for the numerators and scale D for the denominators We start with the fraction which does not contain x

Opposite 6 on scale D set 2 of scale C We do not move the slide Then opposite 4 on scale D we find 133 on scale C We estimate the number to be *more than* 1 and *less than* 10 Then $x = 1\ 33$

The explanation for this procedure is as follows We first divide 6 by 2 and get 3 on the D scale To divide 4 by 3 means by what number must we multiply 3 to get 4 We find the answer on the C scale to be 1 33

Have you observed that for any position of the slide in the rule the *ratio of any pair of opposite numbers on scales* C *and* D *is equal to* the ratio of any other opposite numbers on these scales? Now use your slide rule as you study the examples

Example 1 Solve $\frac{3}{10}=\frac{14}{x}$

Solution *Opposite* 10 *on scale* D set 3 on scale C (Use the right index for 10) Opposite 14 on scale C read 467 on scale D When placing the decimal point in 467, note that 10 is about 3 times 3 Then x will be about 3 times 14 Then $x = 46\ 7$

Example 2 Solve $\frac{x}{354}=\frac{7\ 61}{84}$

Solution Opposite 84 on D set 761 on C Opposite 354 on D read 321 on C Since 7 61 is about $\frac{1}{10}$ of 84, x is about $\frac{1}{10}$ of 354

Then $x = 32\ 1$ Notice that the numerators are read on scale C

Example 3. Solve $\frac{6}{x}=\frac{17}{45}$.

Solution Place 17 on scale C opposite 45 on scale D Since 6 on scale C extends beyond the D scale, we interchange 1 L and 1 R We do this by placing the hairline over 1 L of scale C

and then moving the slide until 1 R is under the hairline Now opposite 6 on scale C, we read 159 on scale D We place the decimal point by inspection Then $x = 15.9$

EXERCISES [A]

Solve, using the slide rule

1 $\frac{x}{75} = \frac{40}{55}$
3 $\frac{x}{435} = \frac{532}{64}$
5 $\frac{81}{x} = \frac{57}{31}$
2 $\frac{x}{38} = \frac{84}{25}$
4 $\frac{3x}{11.3} = \frac{14}{43}$
6 $\frac{4x}{39} = \frac{125}{76}$

THE BINARY SYSTEM OF REPRESENTING NUMBERS (OPTIONAL)

Before we discuss the binary system, let us examine the structure of the decimal system, which we use constantly

You know that

$$\begin{aligned} 13 &= 1 \text{ ten} + 3 \text{ units} \\ &= 1(10)^1 + 3(10)^0, \\ 276 &= 2 \text{ hundreds} + 7 \text{ tens} + 6 \text{ units} \\ &= 2(10)^2 + 7(10)^1 + 6(10)^0, \\ \text{and} \quad 4802 &= 4 \text{ thousands} + 8 \text{ hundreds} + 0 \text{ tens} + 2 \text{ units} \\ &= 4(10)^3 + 8(10)^2 + 0(10)^1 + 2(10)^0 \end{aligned}$$

From these examples we understand that the decimal system of representing numbers has 10 for its base We also see that the absolute value of each digit is affected by its *place* in the number For example, the 7 in 276 means 7 tens, or $7 \times (10)^1$ Do you see that the expressions 13, 276, and 4802 are convenient ways of denoting the sums of powers of 10?

Other numbers, such as 2, 8, and 12 can be used as bases for number systems Why do you think we use 10 as a base? Can you give a reason for using 12 as a base instead of 10?

The binary system of representing numbers is a short way of representing numbers as the sums of powers of 2 It has the base 2 and uses only two symbols, 1 and 0 The table below shows how the integers one through ten are written in the decimal and binary systems

Decimal Notation	1	2	3	4	5	6	7	8	9	10
Binary Notation	1	10	11	100	101	110	111	1000	1001	1010

Because the binary system uses only two symbols, 1 and 0, it is used in some of the large electrical computers. The 1 can be represented by an open circuit, a magnetized spot on a drum, or a punched hole in a tape, while the 0 can be represented by the opposite in each case.

Now study the meaning of some binary numbers

10 means $1(2) + 0(2)^0$
111 means $1(2)^2 + 1(2)^1 + 1(2)^0$
1011 11 means $1(2)^3 + 0(2)^2 + 1(2)^1 + 1(2)^0 + 1(2)^{-1} + 1(2)^{-2}$

Now study Examples 1 and 2 which illustrate how a number expressed by the binary system can be written by the decimal system

Example 1 Change 111_2 to the decimal scale

Solution $111_2 = 1(2)^2 + 1(2)^1 + 1(2)^0$
$= 4 + 2 + 1$
$= 7$

Then $111_2 = 7_{10}$

Example 2. Change $1101\ 1_2$ to the decimal scale

Solution $1101\ 1_2 = 1(2)^3 + 1(2)^2 + 0(2)^1 + 1(2)^0 + 1(2)^{-1}$
$= 8 + 4 + 0 + 1 + 5$
$= 13\ 5$

Then $1101\ 1_2 = 13\ 5_{10}$

EXERCISES

Change from the binary system to the decimal system

1. 10_2	3. 1_2	5 1110_2	7. 1111_2	9 $111\ 1_2$
2 111_2	4. 11_2	6. 1010_2	8 11110_2	10. $101\ 1_2$

Now study Examples 3 and 4 to learn how a number written in the decimal system can be written in binary form

Example 3. Change 19_{10} to the binary form

Solution 1 The largest power of 2 in 19 is 16 or 2^4 $19 - 16 = 3$
The largest power of 2 in 3 is 2^1 or 2 $3 - 2 = 1$ $1 = 2^0$

$$2^4 = 10000$$
$$2^1 = 10$$
$$2^0 = 1$$

Then $19_{10} = 10011_2$

Solution 2

$$
\begin{array}{r|l}
2 & 19 \\ \hline
2 & 9 \text{ R } 1 \\ \hline
2 & 4 \text{ R } 1 \\ \hline
2 & 2 \text{ R } 0 \\ \hline
2 & 1 \text{ R } 0 \\ \hline
 & 0 \text{ R } 1
\end{array}
$$

By reading the remainders upward we find the number to be 10011

Then $\qquad 19_{10} = 10011_2$

Example 4 Change 7 to the binary scale

Solution $\quad 7 = 1\ 4$ halves

$$
\begin{aligned}
4 \text{ halves} &= 8 \text{ fourths} \\
8 \text{ fourths} &= 1\ 6 \text{ eighths} \\
6 \text{ eighths} &= 1\ 2 \text{ sixteenths}
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{1}{2} + 0 + \tfrac{1}{8} + \tfrac{1}{16} \\
&= 1(2)^{\ 1} + 0(2)^{\ 2} + 1(2)^{\ 3} + 1(2)^{-4} \\
&\quad\ 1 \qquad\quad 0 \qquad\quad 1 \qquad\quad 1
\end{aligned}
$$

Then $7_{10} = 1011_2$ to four places

EXERCISES

Change from the decimal scale to the binary scale

1 7	4 10	7 8	10 33	13 6
2 4	5 15	8 16	11 100	14 13
3 9	6 24	9 32	12 415	15 24

Now study Examples 5 and 6 to learn how to add and subtract with the binary scale Remember that $1 + 1 = 10$ in the binary scale

Example 5 Add 101011 and 11001

Solution

$$
\begin{array}{r}
101011 \\
\underline{11001} \\
1000100
\end{array}
\text{, the sum}
$$

Example 6 Subtract 11110 from 101010

Solution

$$
\begin{array}{r}
101010 \\
\underline{11110} \\
01100
\end{array}
\text{ the difference}
$$

EXERCISES

Add or subtract, as indicated

1 111 + 101　　4 1111 + 111 1　　7 1110 − 111
2 111 + 110　　5 1110 + 1010　　8 1000 − 10 1
3 10101 + 1010　　6 1011 − 1010　　9 101 1 − 1 1

If you can add and subtract with the binary scale you can easily multiply and divide with it

Example 7. Multiply 110 by 111

Solution

```
     110
     111
    ----
     110
    110
   110
  ------
  101010, the product
```

Example 8 Divide 100011 by 111

Solution

```
        101, the quotient
  111)100011
      111
      ---
       11
       00
       ---
       111
       111
       ---
```

EXERCISES

Multiply or divide, as indicated

1 11 × 11　　4 1111 × 11　　7 101101 ÷ 1001
2 100 × 11　　5 1110 × 101　　8 100111 ÷ 1101
3 100 × 100　　6 10 1 × 101 1　　9 1111 1 ÷ 10 1

Checking Your Understanding of Chapter 10

You should now determine whether you have mastered this chapter Be sure that you know

Should you review?

	PAGE
1 The meaning of exact and approximate numbers	285
2 How to round off numbers	285
3 How to compute with approximate numbers	304
4 How to change exponential notation to logarithmic notation, and conversely, to change logarithmic notation to exponential notation	289

5 How to compute with logarithms 305
6 How to solve exponential equations 310
7 How to compute with the slide rule 316–320
8 The meaning and spelling of the following words

MATHEMATICAL VOCABULARY

	PAGE		PAGE
antilogarithm	292	logarithm	287
approximate number	285	logarithmic notation	289
base of a logarithm	288	mantissa	291
characteristic	291	rounding off numbers	285
exponential equation	310	significant figures	286
exponential notation	289	unit of measure	286
interpolation	300		

[A]

Round off the following numbers to the nearest tenth.

1 7 46   2. 31 94   3. 81 35

4. What is the number of significant figures in the measurement 13 02 in ? in the measurement 40 0 inches?

Exercises 5–25 are to be solved without reference to a table of logarithms

Give the characteristic of the logarithm of each of the following numbers

5. 1 003   6. 72 9   7. 7200   8. 0 004   9. 0 0763

The mantissa of the logarithm of 8364 is 9224 Find the numbers whose logarithms are

10. 1 9224   13. 8 9224 − 10   16. 17 9224 − 20
11. 3 9224   14. 9 9224 − 10   17. 4 9224
12. 0 9224   15. 2 9224   18. 6 9224

19. The mantissa of log 802 is 9042, and the mantissa of log 803 is 9047. Find the mantissa of log 8026

If $\log 6 = 0\ 7782$ and $\log 2 = 0\ 3010$, find

20 log 12   22. log 4   24. log $\frac{1}{3}$
21 log 3   23. log 216   25 log 300

Use a table of logarithms to make the following computations:

26. 85×40
27. $745 \div 1.32$
28. 472^2
29. $(1.09)^5$
30. $\sqrt{0.645 \times 0.95}$
31. $\dfrac{18.37 \times 0.065}{43.6}$
32. $\sqrt[3]{0.0986}$
33. $\sqrt[4]{4.127}$
34. $1.034^3 \times 4.78$
35. $\sqrt{7.146 \times 84}$
36. $364.8 \div 4.5^3$
37. $\sqrt{\dfrac{4.61}{1.84 \times 2.07}}$
38. Solve $3^{x+1} = 54$
39. Solve $7.2^x = 60$

40. Find the compound amount of \$1675 invested at 3% for 8 years when the interest is compounded annually

41. Change $\log_4 25$ to exponential form

(A) CUMULATIVE REVIEW

If $a = 4$, $b = -5$, and $c = 2$, find the value of

1. $a^3 + b^3$
2. $b^3 - 2a^3$
3. $10 + 2ab$
4. $bc + 10$
5. $abc + 60$
6. $a^0 + b^2 + c^3$

Factor

7. $9x^2 - 25$
8. $x^6 - y^6$
9. $x^3 + 27$
10. $ax^2 - ay^2$
11. $x^2 - 4xy + 4y^2 - c^2$
12. $x^3 + x - 10$

13. Find the value of $64^{-\frac{2}{3}} + 15^0$

14. Find the fourth proportional to 6, 8, and 12

15. Find the ratio of

a. 7 to 21 b. 20 to 4 c. ab to b^2 d. πd to d

16. If 280 feet of fence cost \$6960, what will 160 feet of the fence cost?

17. Combine $\dfrac{y}{y-2} + \dfrac{y}{y+2} + \dfrac{y^2}{4-y^2}$.

18. Solve $W = P\dfrac{2R}{R-r}$ for R

19. The sum of three consecutive even integers is 138. Find the integers

20. Find to the nearest hundredth the value of $\frac{2}{3} - \sqrt{\frac{1}{3}}$

21 $x^n(x^n + 2x^{-n}) =$?

22 Solve $A = p(1 + rt)$ for r

23 Solve $\sqrt{2x + 3} = 6$

24 Simplify $\sqrt{300} + \sqrt{75} - \sqrt{48}$

25 Find the slope and y-intercept of the graph of the equation $3y = 5 - 6x$

26 $V = \frac{1}{6}\pi d^3$ is a formula for finding the volume of a sphere How is the volume of a sphere affected when its diameter is cubed?

[B]

27 Solve $ax - a^2 = b(x - 3a + 2b)$ for x

28 Factor $x^4 + 2x^2y^2 + 9y^4$

29 Graph the function $y = x^2 - 8x$

30 A number consists of two digits The number exceeds 8 times the sum of its digits by 7, and if 54 is subtracted from the number, the digits are reversed Find the number

31 The sum of the ages of a father and son is 55 years If the age of the son is doubled, it will be three fourths of the father's age How old is each?

32 Find the equation of the line that passes through the points $(-3\ 2)$ and $(5, -4)$

33 Find three numbers such that their sum is 43, the first is $\frac{2}{3}$ of the second and the third exceeds the second by 3

CHAPTER TESTS

[Test A]

1 Give the characteristic of the logarithm of each number

a 425 6 b 0 013

2 The mantissa of the logarithm of 596 is 7752 Find the numbers whose logarithms are

a 2 7752 b 9 7752 − 10

3 Change $\log_4 64$ to exponential form

4 Change $2^x = 8$ to logarithmic form

5 Find the value of $\log_5 125$

6 Solve $2x^5 = 810$

7. Find the product of the approximate numbers 18.43 and 1.32

8 Complete $\log x + \log y = \log$ __?__

9. Complete $3 \log m = \log$ __?__

Use a log table to find the results of the following

10 $\dfrac{78.54 \times 125}{0.42}$

11. $\sqrt[5]{42.3}$

12 $\frac{4}{3} \times 3.14 \times 17^3$

13 $1456 \div 62.5$

14. $A = P(1 + r)^n$ Find the amount of \$750 compounded annually at 3% at the end of 8 years

15 $V = \sqrt{2gs}$ Find V when $g = 32.2$ and $s = 435$

16 If the logarithm of a number is 3.1472, what is the logarithm of a number 1000 times as large?

[Test B]

Do exercises 1–10 without the use of tables

1 Solve $3^x = 243$

2 Solve $8^{x-2} = 16$

3. Solve $5^{3x} = 15625$

4. Solve $2^y = \frac{1}{128}$

5. Find $\log \sqrt[3]{2}$ if $\log 2 = 0.3010$

6 $\log 70 = 1.8451$ and $\log 14 = 1.1461$ Find $\log \sqrt{5}$

7 Does $\dfrac{\log 5m}{\log 2m} = \dfrac{2}{5}$?

8 Express as a single logarithm

a $\log x + \log y$

b $\log x^3 - \log x$

9 If $\log 3 = 0.4771$ and $\log 2000 = 3.3010$, find $\log 60$

10 $\log 672 = 2.8274$ Does antilog $8.8274 - 10 = 6.72$?

Compute with logarithms, using a table

11. $\sqrt[5]{478.3}$

12. $76.43^{-\frac{1}{3}}$

13 Find $\log_8 400$

14 $\sqrt{2.163 \times 14.7}$

15. $\sqrt[3]{0.0008603}$

16 Find $\log_6 186$

17 Find log 45 to base 8

CHAPTER 11

Trigonometry

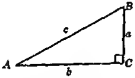
In this chapter you will study principles of indirect measurement

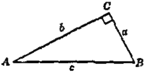
The word trigonometry is derived from two Greek words meaning "triangle measurement." In fact, trigonometry had its origin in the finding of distances and angles, which is still one of the chief purposes of elementary trigonometry. It includes a study of the relation between the angles and the sides of a triangle. Trigonometric functions and their relations with other functions are used extensively in the application of mathematics to science and engineering.

If you have studied trigonometric ratios in elementary algebra or plane geometry, you are familiar with the discussion that follows.

Trigonometric Ratios[A]

It is customary, in the study of trigonometry, to let the capital letters A, B, and C represent the angles of a triangle, C being a right angle. We then denote the sides of the triangle by the lower-case letters a, b, and c in such manner that each angle and the side opposite it use the two forms of the same letter. We call such a triangle a standard right triangle.





In any right triangle,

1 The sine (sin) of an acute angle is the ratio of the opposite side to the hypotenuse.

2 The cosine (cos) of an acute angle is the ratio of the adjacent side to the hypotenuse.

3 The tangent (tan) of an acute angle is the ratio of the opposite side to the adjacent side.

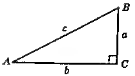
These three ratios, sine, cosine, and tangent, are called the principal trigonometric ratios and are the only ones considered in this book. There are three other trigonometric ratios used in trigonometry. These are the reciprocals of the sine, cosine, and tangent. On the next page the three principal trigonometric ratios are defined in terms of the standard right triangle.

In a standard right triangle

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}, \quad \sin B = \frac{b}{c}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}, \quad \cos B = \frac{a}{c}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}, \quad \tan B = \frac{b}{a}$$

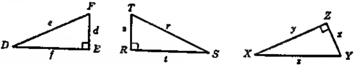


These definitions should be memorized so that the ratios can be read from a figure instantly

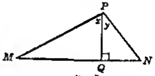
ORAL EXERCISES

[A]

1 Using the three triangles below, give the sine, cosine, and tangent of each acute angle in terms of the sides of the right triangle Thus $\sin D = \frac{d}{e}$, and so on

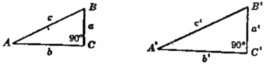


2 In $\triangle MNP$, $PQ \perp MN$ Give the three trigonometric ratios for angles M, x, y, and N in terms of MP, MQ, PQ, PN, and QN



Trigonometric Functions [A]

As long as an acute angle remains constant, the values of its sine, cosine and tangent ratios remain constant To demonstrate this we shall let ABC and $A'B'C'$ be two right triangles having $\angle A = \angle A'$



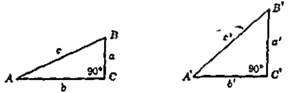
The triangles are similar because two right triangles having an acute angle of one equal to an acute angle of the other are similar

Then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ because corresponding sides of similar triangles are in proportion From this it follows that $\frac{a}{c} = \frac{a'}{c'}$ Why? But $\frac{a}{c} = \sin A$, and $\frac{a'}{c'} = \sin A'$ Consequently, $\sin A = \sin A'$

In similar manner it can be shown that as long as an acute angle remains constant, its cosine and its tangent remain constant

Conversely, it can be shown that as long as the sine, cosine, and tangent of an acute angle remain constant, the size of the angle remains constant

As an acute angle changes in size, its sine, cosine, and tangent change To demonstrate this we shall let ABC and $A'B'C'$ be two right triangles in which angle A is not equal to angle A'



If we suppose for the moment that $\sin A = \sin A'$, we have $\frac{a}{c} = \frac{a'}{c'}$ If this is true, $\frac{a}{a'} = \frac{c}{c'}$ Why? Then the triangles are similar, because two right triangles having the hypotenuse and the leg of one proportional to the hypotenuse and the leg of the other are similar But since corresponding angles of similar triangles are equal, $\angle A = \angle A'$, which is contrary to our original statement that $\angle A$ is not equal to $\angle A'$ Therefore we know that two unequal angles do not have equal sines

In similar manner it may be shown that as an angle changes its size, its cosine and tangent change

Conversely, it may be shown that as the sine, cosine, and tangent change, the acute angle changes in size

Since the sine, cosine, and tangent depend upon the size of an angle for their values, we say that they are functions of the angle, that is, trigonometric functions

Tables of the Trigonometric Functions[A]

The values of the trigonometric functions for different angles have been calculated by methods of more advanced mathematics and have

been collected in tabular form known as tables of trigonometric functions In Table III at the end of the book the values of the principal functions from 0° to 90° are given to 4 decimal places For angles between 0° and 45° read the angles in the *left* column and use the column designations sin cos tan at the top For the angles between 45° and 90° read the angles in the *right* column and use the column designations tan sin cos at the bottom

EXERCISES

[A]

Using Table III find the value of each of the following functions

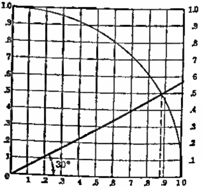
1 cos 5°
2 sin 6° 10
3 tan 11° 20
4 sin 21° 50
5 cos 32° 40
6 tan 39° 30
7 tan 46° 10
8 sin 49° 20
9 cos 82° 20
10 cos 62° 30
11 sin 54° 50
12 tan 76° 10

13 As the angles increase which of the three functions increase and which decreases?

Find the size of the angle in each of the following

14 sin A = 0843
15 cos A = 9932
16 tan A = 1733
17 sin A = 4067
18 tan A = 6873
19 cos A = 6905
20 sin A = 7771
21 cos A = 5736
22 tan A = 1 935
23 sin A = 8923
24 cos A = 2391
25 tan A = 17 17

26 Tell how we may find the trigonometric functions of an angle of 30° with this drawing



Interpolation[A]

In order to find the value of a function of an angle not given in the table, it is necessary to use interpolation

Example 1. Find sin 32° 12′

Solution Since 12′ is $\frac{2}{10}$ of the way between 10′ and 20′, the sine of 32° 12′ is found by adding to the sine of 32° 10′ two tenths of the difference between sin 32° 10′ and sin 32° 20′

Thus
$$\begin{array}{rr} \sin 32° 20' = & 5348 \\ \sin 32° 10' = & 5324 \\ \hline \text{Tabular difference} = & 24 \end{array}$$

Think "$\frac{2}{10}$ of 24 is 5," and mentally add 5 to the last digit of 5324, which gives 5329 Then sin 32° 12′ = 5329

In the case of increasing functions, such as the sine and tangent, the correction must be added to the value of the function of the smaller angle, but in a decreasing function, such as the cosine, the correction must be subtracted

Example 2 Find cos 53° 37′

Solution
$$\begin{array}{rr} \cos 53° 30' = & 5948 \\ \cos 53° 40' = & 5925 \\ \hline \text{Tabular difference} = & 23 \end{array}$$

$\frac{7}{10}$ of 23 is 16 Subtracting 16 mentally from the last two figures of 5948 gives 5932 Then cos 53° 37′ = 5932

[A] **EXERCISES**

Find the value of

1. sin 36° 15′	6 cos 67° 12′	11 sin 82° 54′
2. sin 45° 24′	7 tan 11° 17′	12. cos 7° 19′
3. sin 72° 42′	8 tan 49° 41′	13 cos 42° 18′
4 cos 8° 16′	9. tan 86° 4′	14. tan 12° 6′
5 cos 36° 23′	10 sin 3° 9′	15 tan 63° 36′

Inverse Use of Tables[A]

The process of finding the angle when the value of the function is given will be illustrated by two examples

Example 1. Find B if tan B = 2905

Solution The values of the tangents of angles between 0° and 45° are less than 1, and of angles between 45° and 90°, greater than 1 Looking in the column headed "tan" at the top, we find that .2905 lies between the values 2899 and .2931 in the table

Then

$$\begin{array}{r} 2931 = \tan 16° 20' \\ 2899 = \tan 16° 10' \\ \hline 32 = \text{tabular difference} \end{array}$$

The difference between 2905 and 2899 is 6 $\frac{6}{32}$ or $\frac{3}{16}$ of $10' = 2'$, the amount to be added to 16° 10′ Therefore $B = 16° 12'$

Example 2 Find B if $\cos B =$ 3140

Solution The values of the cosines of angles between 0° and 45° decrease from 1 to about 7, and between 45° and 90° they decrease from about 7 to 0 Looking in the column designated cos at the bottom we find that 3140 lies between the values 3145 and 3118 in the tables

Then

$$\begin{array}{r} 3145 = \cos 71° 40' \\ 3118 = \cos 71° 50' \\ \hline 27 = \text{tabular difference} \end{array}$$

The difference between 3140 and 3118 is 22 Then $\frac{22}{27}$ of $10 = 8$ the amount to be subtracted from 71° 50 Then $3140 = \cos 71° 42$

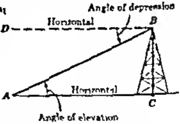
Since the cosine is a decreasing function the difference is subtracted from 71° 50

EXERCISES (A)

Determine the angle

1 $\sin A =$ 0730	5 $\sin A =$ 4225	9 $\tan S = 1\ 387$
2 $\cos B =$ 9912	6 $\sin B =$ 5737	10 $\tan A = 2.188$
3 $\tan C =$ 1600	7 $\cos t =$ 7840	11 $\sin Y =$ 9054
4 $\tan X =$ 4000	8 $\cos C =$ 7000	12 $\cos B =$ 0065

Angles of Elevation and Depression (A)



In the figure $\angle A$ is the angle of elevation of B from A $\angle ABD$ is the angle of depression of A from B Why does $\angle ABD = \angle CAB$? Notice that the angle of elevation and the angle of depression are measured from the horizontal

Finding Distances and Angles, Using Trigonometric Functions (A)

In finding distances and angles by means of trigonometric functions the following directions will be helpful

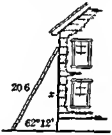
In Using Trigonometric Functions to Find Distances and Angles,

1. Draw a fairly accurate figure
2. Place the known values on the figure
3. Represent by a letter the quantity to be found
4. Write the proper equation
5. Solve the equation and check.

Example 1 A ladder leans against a building and mal an angle of 62° 12′ with the ground If the ladder is 2(feet long, at what height on the building does it touch?

Solution Draw a figure and label it

$$\frac{x}{20.6} = \sin 62° 12'$$
$$x = 20.6 \sin 62° 12'$$
$$x = 20.6(.8846)$$
$$x = 18.22276 \text{ or } 18.22$$



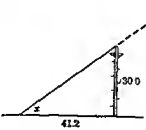
The ladder reaches a point 18.22 feet high on the wall We round the answer off to 4 significant figures, since the tables are 4 place The answer cannot have a greater degree of accuracy than the tables Since the ladder is measured to three significant figures, we round off our answer to 18.2 because it cannot have a greater degree of accuracy than the least accurate of the given data, which are considered approximate

Example 2 Find the angle of elevation of the sun when a telephone pole whose height is 30.0 feet casts a shadow 41.2 feet in length

Solution Draw a figure and label it

$$\tan x = \frac{30}{41.2}$$
$$\tan x = .7282^*$$

From the table,
$$x = 36° 4', \text{ or } 36°$$



*This equation is not algebraic, It is the simplest type of trigonometric equation, which is one of many types of equations in mathematics called *transcendental equations*

EXERCISES

[A]

In a right triangle

1 Given $A = 60° 15'$, $b = 4\ 000$, find c

2 Given $A = 12° 30'$, $b = 4\ 00$, find a

3 Given $B = 56° 10'$, $b = 10\ 6$, find a

4 Given $A = 20° 50'$, $c = 80\ 6$, find a

5 Given $B = 61° 13'$, $c = 16\ 12$, find a

6 Given $a = 4\ 84$, $b = 3\ 63$, find A

7 Given $a = 6\ 52$, $c = 9\ 84$, find B

8 A man standing 110 feet from the foot of a chimney finds that the angle of elevation of the top of the chimney is 47° 25' Find the height of the chimney

9 A rectangle 18 5 feet long has a diagonal 21 2 feet long What angles does the diagonal make with the sides of the rectangle?

10 From the top of a building 60 00 fret high, the angle of depression of an automobile on a road is 24° 18 How far is the automobile from the foot of the building?

11 At a horizontal distance of 112 5 feet from the base of a tower, the angle of elevation of the top is 69° 24' Find the height of the tower

12 The Washington Monument is 555 feet high What is the angle of elevation of the top when viewed from a point that is a half mile away and on a level with the foot of the monument?

13 At the time when the angle of elevation of the sun is 39° 10, the length of the shadow of a tree is found to be 76 3 feet What is the height of the tree?

14 A line joins a vertex of a square to the midpoint of an opposite side Find the acute angles the segment makes with three sides of the square

15 On a certain grade a railroad track rises 1 foot for each 32 25 feet measured along the track What angle does the track make with the horizontal?

16 A mine shaft extends downward at an angle of 6° 27′ with the horizontal How long will the shaft have to be in order to reach a point 60 feet vertically below the surface?

Logarithms of Trigonometric Functions[A]

The logarithms of the values of the trigonometric functions are given in Table IV The column designations are "log sin," "log cos," and "log tan "

To avoid the printing of negative characteristics, the number 10 has been added to each logarithm with the exception of those of tangents of angles equal to or greater than 45° Hence in writing down any logarithm from this table with the exceptions just noted, − 10 should be written after it

[A] **EXERCISES**

Find each of the following logarithms

1 log sin 12° 30
2 log cos 15° 10′
3 log tan 22° 40
4 log sin 14° 8
5 log cos 21° 17′
6 log tan 33° 53′
7 log sin 38° 19′
8 log cos 42° 14′
9 log tan 19° 35′
10 log sin 51° 42′
11 log cos 56° 15′
12 log tan 61° 18′
13 log sin 72° 54′
14 log cos 75° 27′
15 log tan 82° 11′
16 log sin 88° 21′
17 log cos 87° 3′
18 log tan 45°

Find A corresponding to each of the following logarithms

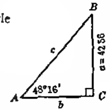
19 log sin A = 7 9408 − 10
20 log cos A = 9 9994 − 10
21 log tan A = 8 9420 − 10
22 log sin A = 9 1466 − 10
23 log cos A = 9 9932 − 10
24 log tan A = 9 7236 − 10
25 log tan A = 10 1085 − 10
26 log cos A = 9 7781 − 10
27 log sin A = 9 9138 − 10
28 log sin A = 9 9574 − 10
29 log cos A = 9 6083 − 10
30 log tan A = 10 5453 − 10
31 log tan A = 10 7592 − 10
32 log cos A = 9 1953 − 10

Solution of Right Triangles by Logarithms[A]

Solving a triangle means finding the remaining parts of the triangle when certain parts are known You will save time and avoid errors

if you first select all the formulas to be used and make a complete outline of the work to be performed

Example 1. Solve the right triangle ABC, given $A = 48° 16'$ and $a = 42.56$



Solution. Draw and label the figure

1 To find B

$$B = 90° - 48° 16' = 41° 44'$$

Outline

2 To find c

$$\frac{42.56}{c} = \sin 48° 16'$$

$$c = \frac{42.56}{\sin 48° 16'}$$

$$\begin{array}{r} \log 42.56 = \\ \underline{\log \sin 48° 16' =} \\ \log c = \\ c = \end{array}$$

3 To find b

$$\frac{42.56}{b} = \tan 48° 16'$$

$$b = \frac{42.56}{\tan 48° 16'}$$

$$\begin{array}{r} \log 42.56 = \\ \underline{\log \tan 48° 16' =} \\ \log b = \\ b = \end{array}$$

Now we are ready to fill in the outline Note the following steps

FIRST Turn to Table II, look up the logarithm of 42.56, which is 1.6290, and place it in both columns of the outline, as shown below

SECOND Turn to Table IV, look up 48° 16', find $\log \sin 48° 16' = 9.8729 - 10$, and place it in its proper place, then find $\log \tan 48° 16' = 10.0496 - 10$, and place it in its proper place

THIRD Subtract in both columns to find $\log c$ and $\log b$

FOURTH Turn to Table II to find c and b

Completed outline and solution

2 To find c

$$\frac{42.56}{c} = \sin 48° 16'$$

$$c = \frac{42.56}{\sin 48° 16'}$$

$$\begin{array}{r} \log 42.56 = 1.6290 \\ \underline{\log \sin 48° 16' = 9.8729 - 10} \\ \log c = 1.7561 \\ c = 57.03 \end{array}$$

3 To find b

$$\frac{42.56}{b} = \tan 48° 16'$$

$$b = \frac{42.56}{\tan 48° 16'}$$

$$\begin{array}{r} \log 42.56 = 1.6290 \\ \underline{\log \tan 48° 16' = 10.0496 - 10} \\ \log b = 1.5794 \\ b = 37.97 \end{array}$$

CHECK First compare the reasonableness of the results with the diagram If we have made a large error, it will be detected For an accurate check, use one of the ratios not used before, or the Pythagorean relation from geometry that $a^2+b^2=c^2$ We can change the formula $a^2+b^2=c^2$ into an equivalent formula adaptable to the use of logarithms, as follows

$$a^2+b^2=c^2$$
$$a^2=c^2-b^2$$
$$a^2=(c+b)(c-b)$$

Then $2\log a=\log(c+b)+\log(c-b)$

Does $2(1\ 6290)=1\ 9777+1\ 2801$?

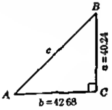
Does $3\ 2580=3\ 2578$? Yes

$$\begin{aligned} c &= 57\ 03 \\ b &= \underline{37\ 97} \\ c+b &= 95\ 00 \\ c-b &= 19\ 06 \end{aligned}$$

Notice that the difference of the two numbers in the last step of the check is only 0 0002

Example 2 Solve the right triangle ABC, given $a=40\ 24$ and $b=42\ 68$

Solution Draw and label the figure



1 To find A

$$\tan A=\frac{40\ 24}{42\ 68}$$

$\log 40\ 24=1\ 6046$

$\log 42\ 68=1\ 6302$

$\log\tan A=9\ 9744-10$

$A=43°\ 19'$

2 To find c

$$\frac{40\ 24}{c}=\sin A$$

$$c=\frac{40\ 24}{\sin A}$$

$\log 40\ 24=1\ 6046$

$\log\sin A=9\ 8364-10$

$\log c=1\ 7682$

$c=58\ 64$

3 To find B

$$B=90°-A=90°-43°\ 19'=46°\ 41'$$

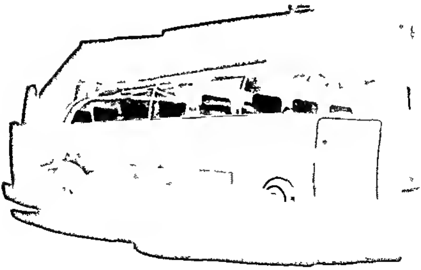
The check is left to the student

THE AUTOMOBILE GAS TURBINE

Do you know what a turbine is? The windmill and waterwheel are turbines which you have probably seen The windmill which has been used on the farm for pumping water receives its power from the wind, and the waterwheel which in pioneer days was used to furnish power for grinding grain and sawing logs and is now used extensively in power plants receives its power from flowing water

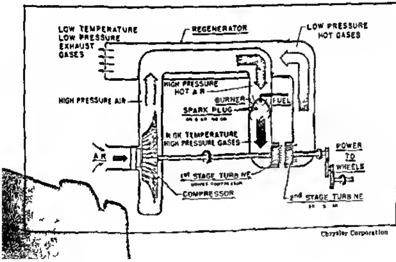
The steam reciprocating engines in the large power plants are gradu ally being replaced by the quieter, more efficient, and longer lasting turbines

You may wonder why the gas turbine has not been used in the automobile Engineers have been trying for years to design an auto mobile gas engine Their aim was to apply the internal combustion principle to the turbine There were many obstacles to be overcome



After much experimentation an experimental gas turbine was finally produced and tested in 1954 The performance of the engine was most gratifying Compared to a piston engine having the same horsepower, the turbine engine has about the same fuel economy, is about 215 pounds lighter, has about one fifth as many moving parts, has no radiator, has only one spark plug, and has better performance

Gas turbines are thus far confined to use in planes and locomotives You may ask "When will *the motorist have a gas turbine in his car?* The cost of producing the motors makes them prohibitive at present, but within a decade they may become feasible It is planned to use such a motor in the *Golden Dolphin* shown in the picture This is a new type of bus designed for a cruising speed of 125 miles per hour



Chrysler Corporation

F C Livingston

CHAPTER 12

Systems of Equations Involving Quadratics

In this chapter we shall study pairs of equations (one, or both, quadratics)

In Chapter 9 you studied quadratic functions of the form $ax^2 + bx + c$ and quadratic equations of the forms $ax^2 + bx + c = 0$ and $y = ax^2 + bx + c$. The first of these two equations contains one variable and the second contains two variables. In this chapter we shall study different forms of quadratic equations in two variables.

The general form of a quadratic equation in two variables is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with at least one of the coefficients A, B, or C not zero. The locus of this equation is the curve of intersection of a plane and a right circular cone. For this reason the curve of a second-degree equation in two variables is called a conic section or simply a conic. The locus may be a circle, an ellipse, a parabola, or a hyperbola, as shown on the next page, or, in special cases, it may be a point, a line, or two lines.

These curves are found frequently in the physical universe. You are already familiar with the numerous applications of the circle. Headlamp and searchlight reflectors, electric-wave reflectors, the path of a projectile fired obliquely upward, an object dropped from a moving conveyance, and the paths of some comets furnish illustrations of the parabola. The track of a plane making an *on pylon* turn in a wind of constant velocity, the orbits of the planets, and the orbit of an electron in reference to the nucleus of the atom of which it is a part, are illustrations of the ellipse. The relation of any two quantities which are inversely proportional is hyperbolic.

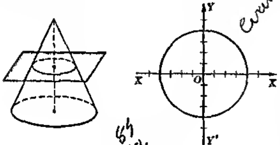
The idea of studying curves by means of algebraic equations was known to the early Greeks. Apollonius is reputed to be the first man to define the circle, parabola, ellipse, and hyperbola as conic sections, though he did not realize that these can be sections of a single cone.

Plane Sections of a Right Circular Cone [H]

The drawings below show how the four conic sections can be formed by a plane intersecting a right circular cone

In a right circular cone, the base is a circle, the axis is the line segment drawn from the vertex to the center of the base, and an element is a line segment drawn from the vertex to any point of the base (circle)

circle



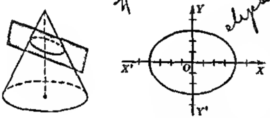
64
60
2/45

The circle $x^2+y^2=16$

The plane section is a circle
The plane is perpendicular to the axis

circle x+y are squared
$x^2+y^2=7$

ellipse

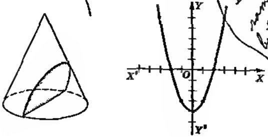


The ellipse $9x^2+16y^2=144$

The plane section is an ellipse
The plane is oblique to the axis

ellipse same as circle but coefficients are different

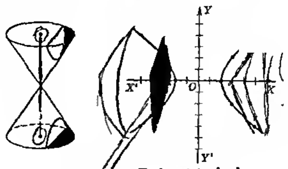
7



The parabola $y=x^2-4$

The plane section is a parabola.
The plane is parallel to an element

The plane section is a hyperbola
The plane is parallel to the axis



The hyperbola $x^2 - y^2 = 4$

How to Graph Quadratic Equations [A]

The graphs of some of the simpler types of quadratic equations in two variables are essential to an understanding of the work of this chapter

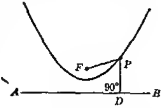
To draw the graph of an equation,

1 Solve the equation for y, unless it is easier to solve for x
2 If the equation is solved for y,
 assign values to x and find the corresponding values of y
 If the equation is solved for x,
 assign values to y and find the corresponding values of x
3 Plot the points and draw a smooth curve through them

The Graph of $y = ax^2 + b$ or $x = ay^2 + b$. The Parabola [A]

In the parabola of the type $y = ax^2 + b$ the axis of the parabola coincides with the y-axis, and the axis of $x = ay^2 + b$ coincides with the x-axis If $a = 0$ in either $y = ax^2 + b$ or $x = ay^2 + b$, the graph is one straight line

The parabola is defined in analytic geometry as the locus of points equidistant from a given point and a given straight line The given point is called the focus of the parabola, and the given line the directrix



Parabola

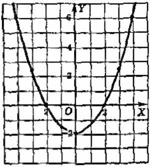
In the figure, F is the focus, AB is the directrix, and P is any point on the parabola By definition, $PF = PD$

Example Construct the graph of $x^2 - 2y = 4$

Solution Solving for y, we have $y = \frac{x^2 - 4}{2}$

In an equation of the type $y = ax^2 + b$ we can construct a better graph if we first find the point where the parabola crosses the y-axis that is, the value of y when $x = 0$ and then assign to x pairs of values having the same absolute value The values chosen here are shown in the table

x	y
0	-2
2 or -2	0
3 or -3	$2\frac{1}{2}$
4 or -4	6



Plotting the points from the table and drawing a smooth curve through them we obtain the graph shown in the figure This graph is symmetrical with respect to the y-axis When $x = ay^2 + b$ the curve is symmetrical with respect to the x axis

EXERCISES

[A]

Construct the graphs of the following equations

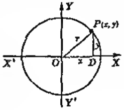
1 $y = 2x^2$ 3 $y^2 - 2x = 4$ 5 $x^2 - 4y = 6$

2 $x = 2y^2$ 4 $y^2 - 4x = 6$ 6 $x^2 - 3y = 8$

The Graph of $x^2 + y^2 = r^2$. The Circle[A]

The graph of every equation of this type is a circle with its center at the origin and its radius equal to r Note that the coefficients of x^2 and y^2 are the same If $r = 0$, the graph is a point This point is called a point circle or a null circle

We know from geometry that a circle is the locus of points at a given distance from a given point, the given point and distance being the center and radius of the circle, respectively

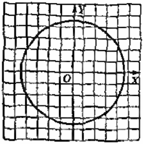


The equation $x^2 + y^2 = r^2$ is easily derived In the figure let the circle have its center at the origin and its radius $= r$ Let $P(x, y)$ be any point on the circle Join P to O and draw $PD \perp OX$ Then $OP = r$, $OD = x$, and

$PD = y$ Then $x^2 + y^2 = r^2$ by the Pythagorean theorem Note that *the circle is symmetric with respect to both axes*

Example Construct the graph of $4x^2 + 4y^2 = 81$

Solution Dividing both members of the equation by 4, we have $x^2 + y^2 = \frac{81}{4}$ The graph of this equation is a circle with its center at the origin and its radius equal to $\sqrt{\frac{81}{4}}$, or $\frac{9}{2}$ It is obviously easier to construct the graph with compasses than to find the points and plot them



(A)

EXERCISES

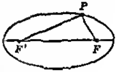
Draw graphs of the following equations

1. $x^2 + y^2 = 25$ 2 $x^2 + y^2 = 36$ 3 $2x^2 + 2y^2 = 9$

The Graph of $ax^2 + by^2 = c$. The Ellipse

The graph of every equation of the form $ax^2 + by^2 = c$, where a, b, and c are positive and $a \neq b$, is an ellipse with its center at the origin If $a = b$, the ellipse becomes a circle The curve is symmetrical with respect to both axes If either a or b is zero, the graph consists of two parallel lines

The ellipse is defined in analytic geometry as the locus of points the sum of whose distances from two given points (the foci) is constant In the figure, if P is any point of the ellipse, $PF' + PF = k$ by definition



Ellipse

Example Construct the graph of $9x^2 + 25y^2 = 225$

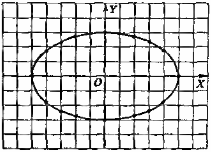
Solution Solving for y, we have $y = \pm\frac{3}{5}\sqrt{25 - x^2}$

By a study of this equation we see that for any value of x greater than 5 or less than -5, y will be imaginary Therefore we shall take values of x from -5 to 5

x	-5	-4	-3	-2	0	2	3	4	5
y	0	± 1.8	± 2.4	± 2.8	± 3	± 2.8	± 2.4	± 1.8	0

(cont on next page)

Plotting the points and drawing the curve we obtain the ellipse shown in the figure



[A]

EXERCISES

Construct graphs of the following equations

1 $4x^2 + 9y^2 = 36$

2 $9x^2 + 4y^2 = 36$

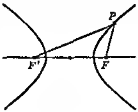
3 $25x^2 + y^2 = 25$

4 $x^2 + 16y^2 = 16$

The Graph of $xy = c$ or $ax^2 - by^2 = c$. The Hyperbola [A]

The graphs of $xy = c$ and $ax^2 - by^2 = c$ where $c \neq 0$ and a and b are positive are hyperbolas If $c = 0$ in either equation, the graph consists of two intersecting straight lines If $xy = 0$, the lines coincide with the axes

In analytic geometry the hyperbola is defined as the locus of points the difference of whose distance from two given points (the foci) is constant The hyperbola is an open discontinuous curve of two branches In the figure F and F' are the foci, and P is any point on either branch of the hyperbola Then $PF' - PF = k$ by definition



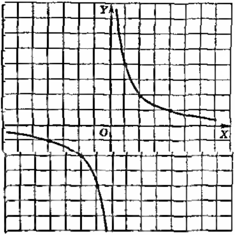
Hyperbola

Example 1 Construct the graph of $xy = 4$

Solution Solving for y, we have $y = \frac{4}{x}$

The substitution of $x = 0$ in this equation involves division by zero However we can say that as x approaches 0, y approaches infinity

x	y
∞	0
4	1
2	2
1	4
0	∞
−1	−4
−2	−2
−4	−1
−∞	−0



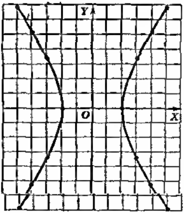
Graphs of the form $xy = c$ approach the axes but do not touch them In the equation $xy = c$, x and y vary inversely Boyle's Law ($pv = k$) is an application of this type of graph

Example 2 Construct the graph of $9x^2 - 4y^2 = 36$

Solution. Solving for y, $y = \pm \frac{3}{2}\sqrt{x^2 - 4}$

From a study of this equation we see that for values of x between 2 and -2, y is imaginary Therefore we shall use $x = 2$ and -2 and other values of x greater than 2 and less than -2

x	y
5	±6.9
4	±5.2
3	±3.4
2	0
−2	0
−3	±3.4
−4	±5.2
−5	±6.9



Graphs of equations of the form $ax^2 - by^2 = c$ are symmetrical with respect to both axes

EXERCISES [A]

Construct graphs of the following equations

1. $xy = 6$
2. $x^2 - y^2 = 16$
3. $y^2 - x^2 = 25$
4. $xy + 8 = 0$
5. $x^2 - 4y^2 = 9$
6. $4x^2 - 9y^2 = 36$
7. $xy = 10$
8. $16x^2 - 9y^2 = 144$

The Graph of a Pair of Straight Lines

If all the terms of a quadratic equation are in the left member and that member is factorable, the graph consists of two straight lines

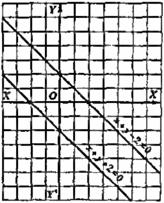
Example Construct the graph of $x^2 + 2xy + y^2 = 4$

Solution $x^2 + 2xy + y^2 - 4 = 0$

Grouping $(x + y)^2 - 4 = 0$

Factoring $(x + y - 2)(x + y + 2) = 0$

or $x + y - 2 = 0$ and $x + y + 2 = 0$



Graphing the linear equations $x + y - 2 = 0$ and $x + y + 2 = 0$ we obtain the two straight lines shown in the figure These two lines are parallel In the case of other equations the lines may intersect or be identical

EXERCISES [A]

Construct the graphs of the following equations

1. $x^2 + xy - 2y^2 = 0$
2. $x^2 - 3xy = 0$
3. $x^2 - 2xy + y^2 = 9$
4. $x^2 - 4xy - 5y^2 = 0$
5. $x^2 - 4y^2 = 0$
6. $xy = 0$
7. $3xy + y^2 = 0$
8. $xy - y^2 - 4x + 4y = 0$

Graphical Solutions of Equations (one, or both, quadratics)[A]

If we plot on the same diagram the graphs of two equations involving quadratics and the two curves intersect, the coordinates of the points of intersection are real roots of the equations. If one equation is linear and one is quadratic, their graphs may have two points, one point, or no point in common. If the graphs have two common coincident points, the line is tangent to the conic.

If the equations are both quadratic, their graphs may intersect in 4 points, 3 points, 2 points, 1 point, or not at all. Such equations may have as many as four real solutions or none. If two of the points of intersection of two curves are coincident, the curves are said to be tangent to each other at the point of coincidence.

Do you see that the graphs of $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$ do not intersect and that they have no common solution?

Example 1 Solve graphically $2x - 3y = 3$ (1)

$xy = 3$ (2)

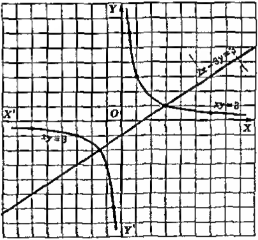
Solution

$xy = 3$

x	y
6	$\frac{1}{2}$
3	1
$\frac{1}{2}$	6
$-\frac{1}{2}$	-6
-3	-1
-6	$-\frac{1}{2}$

$2x - 3y = 3$

x	y
0	-1
$1\frac{1}{2}$	0
3	1



The graph of (1) is a straight line and the graph of (2) is an equilateral hyperbola. The graphs intersect in the points (3, 1) and $(-1\frac{1}{2}, -2)$. Then one common solution is $x = 3$, $y = 1$, and the other is $x = -1\frac{1}{2}$, $y = -2$.

Example 2 Solve graphically $5x + 8y = 40$ (1)

$x^2 + y^2 = 16$ (2)

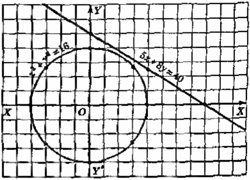
Solution

$5x + 8y = 40$

x	y
0	5
8	0
4	$2\frac{1}{2}$

$x^2 + y^2 = 16$

x	y
0	± 4
± 4	0
3	± 2.6
-3	± 2.6



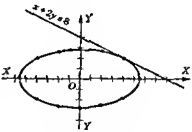
The graph of (1) is a straight line and the graph of (2) is a circle with radius 4 and center at the origin. The graphs do not intersect. Then the two equations do not have any common real solution.

Example 3 Solve graphically $x^2 + 4y^2 = 32$ (1)

$x + 2y = 8$ (2)

Graphic Solution From (1) $y = \pm \frac{1}{2}\sqrt{32 - x^2}$

x	y
0	± 2.8
2	± 2.6
4	± 2
5.6	0
-5.6	0
-4	± 2
-2	± 2.8



The graph of (1) is an ellipse and of (2) a straight line.

From the graph the line is apparently tangent to the ellipse at (4, 2).

The common solutions are (4, 2) and (4, 2), showing that the line is tangent to the ellipse.

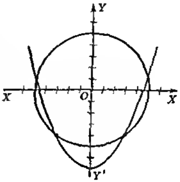
Example 4 Solve graphically $x^2 + y^2 = 25$ (1)

$x^2 - 3y = 21$ (2)

Solution The graph of (1) is a circle whose center is the origin and whose radius is 5. The graph of (2) is a parabola whose turning point is $(0, -7)$. The common solutions are

$(3, -4)$, $(-3, -4)$

$(4.9, 1)$, $(-4.9, 1)$



EXERCISES

[A]

Solve graphically

1. $x^2 + y^2 = 25$
   $x + y = 3$
2. $x^2 - y^2 = 16$
   $x - 2y = -1$
3. $xy = 4$
   $2x - y = 7$
4. $4x^2 + 9y^2 = 25$
   $3x - 2y = 4$
5. $x^2 + y^2 = 34$
   $2x + y = 13$
6. $x^2 + 2y^2 = 8$
   $x + 2y = 4$

[B]

Solve graphically

7. $x^2 + y^2 = 16$
   $9x^2 + 16y^2 = 144$
8. $x^2 + y^2 = 25$
   $xy = 12$
9. $x^2 + y^2 = 36$
   $9x^2 - 16y^2 = 144$
10. $2x^2 - 5y^2 = 27$
    $xy = 18$
11. $x^2 - y^2 = 4$
    $9x^2 + 36y^2 = 16$
12. $x^2 + 3y^2 = 28$
    $xy + 8 = 0$

Algebraic Solution of Pairs of Equations Involving Quadratics [A]

The general solution of pairs of quadratic equations in two variables is complicated and usually leads to solution of higher-degree equations, which we are not prepared to solve at this time. However, the ability to solve three simple cases of sets of equations involving quadratics is sufficient for most applications of mathematics.

Algebraic Solution of One Linear and One Quadratic Equation [A]

Example Solve algebraically $x^2 + 4y^2 = 32$
$x + 2y = 8$

This pair of equations was solved graphically in Example 3, p 354

Solution. The method of the algebraic solution is elimination by substitution We solve the linear equation for one variable and substitute its value in the quadratic equation

Solving $x + 2y = 8$ for x, we have $x = 8 - 2y$

Substituting $x = 8 - 2y$ in $x^2 + 4y^2 = 32$

$$(8 - 2y)^2 + 4y^2 = 32$$
$$64 - 32y + 4y^2 + 4y^2 = 32$$
$$8y^2 - 32y + 32 = 0$$
$$y^2 - 4y + 4 = 0$$
$$(y - 2)(y - 2) = 0$$
$$y = 2 \mid y = 2$$

Substituting these values of y in $x = 8 - 2y$, we have $x = 4$ and 4 This shows that the graphs intersect in two coincident points, (4, 2) and (4, 2), and hence the line is tangent to the ellipse These equations have a pair of solutions, (4, 2) and (4, 2)

[A]

EXERCISES

Solve algebraically

1. $xy = 12$
   $x - y = 1$
2. $x^2 - y^2 = 16$
   $x + y = 8$
3. $x^2 - y = 3$
   $x - y = -3$
4. $3x^2 + xy = 15$
   $2x + y = 2$
5. $3xy + y^2 = 25$
   $x + 2y = 10$
6. $2x^2 - 3xy = 35$
   $3x + 2y = 27$
7. $2x^2 - 3xy + y^2 = 0$
   $5x + y = 12$
8. $x + y = 1$
   $\frac{1}{x} + \frac{1}{y} = 4$
9. $x^2 + 2y^2 = 51$
   $2x + y = 13$
10. $x^2 + y = 34$
    $x - 3y = 12$
11. $2x^2 - 3y^2 = 24$
    $2x - 3y = 0$
12. $x^2 - xy + y^2 = 7$
    $x + y = 4$
13. $4x + 4y - y^2 = 8$
    $2x + y = 8$
14. $2xy + 4x + 3y = 1$
    $2x + y - 3 = 0$

15 $x^2 - 2xy + y^2 = 4$
$2x - y = 3$

16 $xy = 4$
$\frac{1}{x} + \frac{1}{y} = 5$

Algebraic Solution of Pairs of Equations of the Form $ax^2 + by^2 = c$ [A]

Every pair of equations of this form can be solved by the addition-subtraction method or by the substitution method

Example Solve algebraically $x^2 + y^2 = 25$, $x^2 + 4y^2 = 36$

Solution

$$x^2 + y^2 = 25 \qquad (1)$$
$$x^2 + 4y^2 = 36 \qquad (2)$$

S (1) from (2) $\quad 3y^2 = 11$

$$y = \pm \tfrac{1}{3}\sqrt{33}$$

Substituting $y = \frac{1}{3}\sqrt{33}$ in (1), we have

$$x^2 + \tfrac{11}{3} = 25$$
$$3x^2 + 11 = 75$$
$$3x^2 = 64$$
$$x = \pm \tfrac{8}{3}\sqrt{3}$$

Then $(\frac{8}{3}\sqrt{3}, \frac{1}{3}\sqrt{33})$ and $(-\frac{8}{3}\sqrt{3}, \frac{1}{3}\sqrt{33})$ are two solutions

Substituting $y = -\frac{1}{3}\sqrt{33}$ in (1), we have

$$x^2 + \tfrac{11}{3} = 25$$
$$3x^2 + 11 = 75$$
$$3x^2 = 64$$
$$x^2 = \pm \tfrac{8}{3}\sqrt{3}$$

Then $(\frac{8}{3}\sqrt{3}, -\frac{1}{3}\sqrt{33})$ and $(-\frac{8}{3}\sqrt{3}, -\frac{1}{3}\sqrt{33})$ are two more solutions The four solutions are

$\frac{8}{3}\sqrt{3}, \frac{1}{3}\sqrt{33}$ $\qquad$ $\frac{8}{3}\sqrt{3}, -\frac{1}{3}\sqrt{33}$
$-\frac{8}{3}\sqrt{3}, \frac{1}{3}\sqrt{33}$ $\qquad$ $-\frac{8}{3}\sqrt{3}, -\frac{1}{3}\sqrt{33}$

PROOF Does $\frac{64}{3} + \frac{11}{3} = 25$? Yes
Does $\frac{64}{3} + \frac{44}{3} = 36$? Yes

[A] **EXERCISES**

Solve

1. $x^2 + y^2 = 16$
$9x^2 + 25y^2 = 225$

2. $x^2 + 2y^2 = 17$
$2x^2 - 3y^2 = 6$

3 $3x^2 + 4y^2 = 31$
$5x^2 - 6y^2 = 1$

4. $3x^2 + y^2 = 52$
$2x^2 - 5y^2 = 12$

5 $5x^2 - 2y^2 = 13$
$7x^2 - 3y^2 = 15$

6 $4x^2 + 4y^2 = 65$
$6x^2 - 4y^2 = 25$

Many other pairs of equations involving only like terms can be solved either by the substitution method or by the addition subtraction method

Example 2 Solve $2x^2 + 5xy = 33$ (1)

$3x^2 - 2xy = 21$ (2)

Solution M(1) by 2, $4x^2 + 10xy = 66$ (3)

M(2) by 5 $\underline{15x^2 - 10xy = 105}$ (4)

A(3) and (4) $19x^2 = 171$

D_{19} $x^2 = 9$

R_2 $x = \pm 3$

Substituting $x = 3$ in (1) $18 + 15y = 33$

$y = 1$

Substituting $x = -3$ in (1), $18 - 15y = 33$

$y = -1$

Therefore the common solutions are (3 1) and (−3 −1)

PROOF in (1)

$2x^2 + 5xy = 33$

Does $18 + 19 = 33$? Yes

PROOF in (2)

$3x^2 - 2xy = 21$

Does $27 - 6 = 21$? Yes

The proof is the same for either set of roots

EXERCISES [A]

Solve algebraically

1 $2x^2 + 7y = 32$
$3x^2 - 5y = 17$

2 $x^2 + xy = 8$
$3x^2 - xy = 56$

3 $2xy + y^2 = 40$
$3xy + 2y^2 = 68$

4 $x^2 - 3y = 46$
$2x^2 + 3y = 101$

5 $3x + 5y^2 = 44$
$5x - 2y^2 = 32$

6 $5xy + 2y + 60 = 0$
$xy - 3y - 5 = 0$

Algebraic Solution of Pairs of Equations in Which All Terms Containing the Variables Are of the Second Degree [8]

Example Solve algebraically

$$x^2 + 2xy + 3y^2 = 17 \quad (1)$$
$$2x^2 - xy + y^2 = 64 \quad (2)$$

Solution Let $y = vx$ and substitute in both equations

From (1) $x^2 + 2vx^2 + 3v^2x^2 = 17$

Factoring $x^2(1 + 2v + 3v^2) = 17$

and $$x^2 = \frac{17}{1 + 2v + 3v^2} \quad (3)$$

From (2) $\quad 2x^2 - vx^2 + v^2x^2 = 64$

Factoring $\quad x^2(2 - v + v^2) = 64$

and
$$x^2 = \frac{64}{2 - v + v^2} \qquad (4)$$

Equating the values of x^2 from (3) and (4),

$$\frac{17}{1 + 2v + 3v^2} = \frac{64}{2 - v + v^2}$$

Simplifying, $\quad 34 - 17v + 17v^2 = 64 + 128v + 192v^2$

$$17v^2 - 192v^2 - 17v - 128v + 34 - 64 = 0$$
$$-175v^2 - 145v - 30 = 0$$

D_{-5}
$$35v^2 + 29v + 6 = 0$$
$$(5v + 2)(7v + 3) = 0$$

$5v = -2$	$7v = -3$
$v = -\frac{2}{5}$	$v = -\frac{3}{7}$

Substituting these values in (3),

$v = -\frac{2}{5}$	$v = -\frac{3}{7}$
$x^2 = \dfrac{17}{1 - \frac{4}{5} + \frac{12}{25}}$	$x^2 = \dfrac{17}{1 - \frac{6}{7} + \frac{27}{49}}$
$x^2 = 25$	$x^2 = \frac{49}{2}$
$x = \pm 5$	$x = \pm \frac{7}{2}\sqrt{2}$

Since $y = vx$, we find for $v = -\frac{2}{5}$ and $x = \pm 5$, $y = -\frac{2}{5}(\pm 5) = \mp 2$ The $\mp$ sign means that when $x = +5$, $y = -2$, and when $x = -5$, $y = +2$ Hence $(5, -2)$ and $(-5, 2)$ are two sets of roots

For $v = -\frac{3}{7}$ and $x = \pm\frac{7}{2}\sqrt{2}$ $y = vx = -\frac{3}{7}(\pm\frac{7}{2}\sqrt{2}) = \mp\frac{3}{2}\sqrt{2}$ Hence $(\frac{7}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2})$ and $(-\frac{7}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$ are two sets of roots

Therefore the solutions are $(5, -2)$, $(-5, 2)$, $(\frac{7}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2})$, and $(-\frac{7}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$ The student should prove the solutions

The equation $y = vx$ expresses the ratio of y to x This ratio is often used in mathematics to transform expressions into simpler expressions for particular purposes

(3)

EXERCISES

Solve algebraically

1. $x^2 - xy = 48$
   $xy + y^2 = 20$
2. $x^2 + 3xy = 54$
   $xy + 4y^2 = 115$
3. $x^2 + 4y^2 = 13$
   $xy + 2y^2 = 5$
4. $x^2 + xy + 2y^2 = 77$
   $x^2 - xy - 5 = 0$

5. $x^2 - 2xy + 2y^2 = 2$
 $x^2 - y^2 - 3 = 0$

6. $x^2 + xy + y^2 = 52$
 $x^2 - xy + 8 = 0$

7. $x^2 + 2xy - 3y^2 = 5$
 $2x^2 + 5y^2 = 13$

8. $x^2 + 5xy + y^2 = 79$
 $xy = 10$

Special Methods of Solution[8]

Many systems of equations may be solved by other methods. Your success in solving such systems depends on your ingenuity and experience in mathematics. The following examples illustrate typical methods.

Example 1. Solve
$$2x^2 - 3y^2 + 4x + 4y = 55 \quad (1)$$
$$3x^2 + 3y^2 + 6x - 4y = 120 \quad (2)$$

Solution. From a study of the equations we observe that by addition we can eliminate y from the equations and obtain a quadratic equation in x.

A (1) and (2) $\quad 5x^2 + 10x = 175$
$D_5 \quad x^2 + 2x - 35 = 0$
Solving $\quad (x + 7)(x - 5) = 0$
$\quad x = -7$ and $x = 5$

Substituting these values in (1), we obtain

for $x = -7 \quad 98 - 3y^2 - 28 + 4y = 55 \quad (3)$
for $x = 5 \quad 50 - 3y^2 + 20 + 4y = 55 \quad (4)$

Simplifying (3) and (4), we have

from (3) $\quad 3y^2 - 4y - 15 = 0 \quad (5)$
from (4) $\quad 3y^2 - 4y - 15 = 0 \quad (6)$

Since the equations are identical, the solution of either is

$$(3y + 5)(y - 3) = 0$$
$$y = -\tfrac{5}{3}, \text{ and } y = 3$$

The solutions are $(-7, -\frac{5}{3})$, $(-7, 3)$, $(5, -\frac{5}{3})$, and $(5, 3)$.

Example 2. Solve
$$x^2 + y^2 + 4x + 4y = 29 \quad (1)$$
$$xy = -12 \quad (2)$$

Solution. M (2) by 2 $\quad 2xy = -24 \quad (3)$
A (1) and (3), $\quad x^2 + 2xy + y^2 + 4x + 4y = 5$
or $\quad (x + y)^2 + 4(x + y) - 5 = 0$
Factoring $\quad (x + y + 5)(x + y - 1) = 0$
Then $\quad x + y = -5$ and $x + y = 1$

Combining each of these first-degree equations with equation (2), we have two sets

a $x+y=-5$ and *b* $x+y=1$
$xy=-12$ $xy=-12$

These two sets of equations together are equivalent to the original set, that is, their solutions are the solutions of the original set Sets *a* and *b* may be solved by substitution Their solution is left to the student

EXERCISES

Solve algebraically

1. $x^2-16y=0$
$x^2-y^2=64$

2. $4x^2+y^2=52$
$2x^2+y=14$

3. $x-y^2=0$
$x^2-5x+4=0$

4. $x^2+y^2-2x-2y=11$
$xy=2$

5 $x^2+y^2+x+y=14$
$xy=3$

6. $x^2+y^2+2x+6y=-1$
$2x+y+2=0$

7. $x^2+y^2-10x=0$
$x^2+y^2-20y=0$

8 $xy=80$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{8}$

9. $x^2+y^2+4x-6y=39$
$x^2+y^2-14x+6y=-45$

10 $2y^2-9x+8y-19=0$
$4y^2+9x+16y-11=0$

11 $3y^2-16x-30y+75=0$
$x^2+y^2-10y=0$

12. $4x^2+y^2+8x-4y=44$
$y^2-18x-4y-14=0$

13. $x^2-4x+y=0$
$xy-2x-y=0$

14 $x^2-x-y=0$
$2x-y+2=0$

PROBLEMS

[A]

Use two unknowns in the solution of the following problems

1 The sum of two numbers is 5 and the sum of their squares is 13 Find the numbers

2 The difference of two numbers is 5 and the sum of their squares is 53 Find the numbers

3 The difference of two numbers is 5 and their product is 24 What are the numbers?

4. The sum of two numbers is 10 and their product is 21 Find the numbers

5 The difference between the legs of a right triangle is 14 and the hypotenuse is 26 Find the lengths of the sides

6 One train running 5 miles an hour faster than a second train required 4 hours less time to travel 400 miles Find the rate of each train

7 An airplane going a distance of 1000 miles can make the trip in 2 hours less time if it increases its speed 25 miles an hour Find the rate of the plane

8 Two automobiles each travel 72 miles One goes 4 miles an hour faster than the other and makes the run in 12 minutes less time Find the rate of each

9 It takes one automobile 2 hours less time than another to travel a distance of 360 miles If the rate of the first automobile is 6 miles an hour faster than that of the second, find the rate of each automobile

10 The sum of the digits of a two-digit number is 9 Find the number if the sum of the squares of the digits is 53

11 The sum of the squares of the digits of a two digit number is 26 Find the number if the tens digit exceeds the units digit by four

12 The difference of the areas of two squares is 223 square feet and the difference of their perimeters is 24 feet Find a side of each square

13 The difference of the areas of two squares is 1100 square feet and the difference of their perimeters is 40 feet Find a side of each square

14 If the width of a rectangle is increased by 2 inches, the area is 140 square inches If the length is increased by 1 inch the area is 120 square inches Find the dimensions of the rectangle

15 The perimeter of a floor is 44 feet and its area is 120 square feet Find its dimensions

16 It takes 52 rods of fence to enclose a rectangular lot whose area is 1 acre Find the dimensions of the lot (1 acre = 160 sq rd)

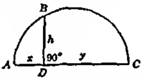
17. The perimeter of a 5-acre rectangular field is 114 rods Find the length and width of the field

18. The ratio of two numbers is 3 to 2, and the difference of their squares is 20 Find the numbers

19 The area of a rectangular field is 40 acres, and the ratio of the width to the length is 2 to 5 Find the length and the width

[B]

20. ABC is a semicircle $BD \perp AC$ From geometry, $h^2 = xy$ Find h, x, and y if $AC = 20$ and $h = 2x$



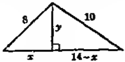
21. The front wheel of a wagon makes 10 revolutions more than the rear wheel in going 360 feet If the circumference of each wheel is increased by 3 feet, the front wheel will make only 6 more revolutions than the rear wheel in going 360 feet What is the circumference of each wheel?

22 At simple interest a sum of money amounted to $2600 for one year, and to $2700 for two years What were the principal and the rate of interest?

23 The amount of a sum of money on interest for 1 year is $4240 If the rate were 1% less and the principal $500 more, the amount would be $4725 Find the principal and the rate

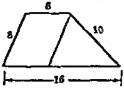
24 The area of a right triangle is 600 square feet and its hypotenuse is 50 feet Find the lengths of the two legs

25 The sides of a triangle are 8, 10, and 14 Find the altitude on side 14



SUGGESTION From the figure, use the Pythagorean Theorem to form two equations

26 The bases of a trapezoid are 6 and 16 inches respectively and the two nonparallel sides are 8 and 10 inches respectively Find the altitude of the trapezoid



SUGGESTION Draw a line between the bases parallel to one of the nonparallel sides as shown in the figure Find the sides and altitude of the resulting triangle

27 An airplane makes a trip of 600 miles and returns Its rate going was $\frac{3}{4}$ the rate returning If it took 1 hour longer to go than to return, what was the rate each way?

28 The altitude of an isosceles triangle is 4 feet and its perimeter is 16 feet Find the sides of the triangle

29 48 pounds of iron are made into a bar If the bar were made 1 foot longer, the weight of a foot would be 1 pound less Find the length of the bar and the weight of 1 foot of the bar

30 Two men can do a piece of work in 6 days It takes one man 5 more days to do the work when working alone than it does the other In how many days can each do the work?

31 The area of a rectangular lot is 60 square rods The diagonal of the lot is 13 rods Find the length and the width of the lot

32 The sides of a triangle are 13, 14, and 15 inches respectively Find the altitude on the longest side and the area of the triangle

33 The sides of an obtuse triangle are 5 inches, 12 inches, and 15 inches. Find the altitude on the shortest side

34 The sides of a triangle are 10, 13, and 18 Find the projections of the first two sides on the third side

35 The sides of a triangle are 7, 25, and 30 inches respectively Find the projections of the other two sides on the short side

[C]

36. A farmer contracted the job of grading a township road. After working alone for 2 days, he hired his neighbor to help him and they completed the work in $5\frac{6}{7}$ days. If the farmer had worked alone, he would have required 4 less days than his neighbor to do the whole job. Find the time each would have required to do the work alone.

37. Bill and Dick start at the same time from the same place. One walks north and the other walks east. At the end of 3 hours Bill has walked 3 miles more than Dick and they are 15 miles apart. What is the rate of each?

38. The fore wheel of a tractor makes 10 revolutions more than the rear wheel when going 360 feet. If the circumference of each wheel were increased by 6 feet, the fore wheel would make only 5 more revolutions than the rear wheel when going this distance. Find the circumference of each wheel.

Checking Your Understanding of Chapter

You should now determine if you have mastered the most important topics of this chapter. Be sure that you know:

	PAGE
1 How to graph quadratic equations	347
2 Why the graphs of second-degree equations are called conic sections	346
3 How to recognize the equation of the parabola, the ellipse, the circle, and the hyperbola	347–351
4 How to find graphically the common solutions of pairs of equations when one, or both, of the equations is a quadratic	353
5 How to solve algebraically pairs of equations when	
(a) one is linear and one is a quadratic	356
(b) both equations are of the form $ax^2 + by^2 = c$	357
(c) all terms containing the variables in both equations are of the second degree	358
6 How to solve verbal problems using pairs of equations, when one or both equations are of the second degree	361–365

Should you review?

CHAPTER REVIEW

[A]

1 Why are the loci of second-degree equations in two variables called conic sections?

2 Define circle parabola ellipse and hyperbola

Construct graphs of the following equations

3 $x^2 + y^2 = 3$	6 $x = 2y^2 - 4$	9 $x^2 - 9y^2 = 9$
4 $y = 4x^2$	7 $x^2 + y^2 = 8$	10 $xy = 6$
5 $x^2 + 9y^2 = 9$	8 $x^2 - 4y^2 = 0$	11 $x^2 - xy - 2y^2 = 0$

Solve graphically and algebraically

12 $x^2 + y^2 = 17$	13 $x^2 + 4y^2 = 20$	14 $x - y = 3$
$x^2 - 4y = 12$	$x - 3y = 1$	$xy = 4$

Solve algebraically

15 $3x^2 - 2y^2 = 30$
$2x^2 + 5y^2 = 77$

16 $7xy - y = 30$
$3xy + 2y = 25$

17 $x^2 - 3xy + y^2 = 31$
$3y - 2x = 5$

18 The sum of two numbers is 11 and the sum of their squares is 65 Find the numbers

19 The area of a right triangle is 60 square feet and the hypotenuse is 17 feet Find the lengths of the two sides

20 The sides of a triangle are 6 8 and 9 Find the altitude on side 6

21 One train running 12 miles an hour faster than a second train required $2\frac{1}{2}$ hours less time to travel 600 miles Find the rate of each train

[A] CHAPTER TEST

1. What type of curve is the graph of each of the following equations?

a $x^2 - y = 6$ b $3x^2 - y^2 = 75$ c $2xy = 7$

d $5x^2 + y^2 = 25$ e $6x^2 + xy - y^2 = 0$

2. Draw a graph of the ellipse $9x^2 + 4y^2 = 36$

3. Draw a graph of the parabola $x + y^2 = 10$

4 Solve graphically $x - 2y^2 = 6$
$x - 4y = 4$

Solve algebraically

5 $y^2 + 6xy = 52$
$y = 2x$

6 $x^2 + 7y = 16$
$3x^2 - 2y = 25$

7 $x^2 + y^2 = 25$
$x - y = 1$

8 The altitude of an isosceles triangle is 3 feet and its perimeter is 18 feet Find the length of each side

9. An airplane makes a trip of 400 miles and returns Its rate going was $\frac{4}{5}$ the rate returning If it took 1 hour longer to go than return, what was the rate each way?

10 The area of a rectangle is 240 square inches and the diagonal is 26 inches Find the dimensions of the rectangle

From this table we see that the coefficient of d in any term is one less than the number of the term Then the eighth term is $a + 7d$, the ninth term is $a + 8d$, and the nth term is $a + (n-1)d$ If we let l denote the nth term or last term, we have

$$l = a + (n-1)d$$

Example 1 Find the 14th term in the sequence 5, 7, 9, 11,

Solution In this example $a = 5$, $d = 2$ and $n = 14$

$$l = a + (n-1)d$$
$$l = 5 + (14-1)2$$
$$l = 5 + 26$$
$$l = 31$$

EXERCISES

[A]

1 Find the 11th term in the sequence 3, 4, 5,

2 Find the 20th term of the sequence 1, 3, 5,

3 Find the 10th term of the sequence 17, 13, 9,

4 Find the last term of the sequence $-2, 0, 2, 4,$ to 24 terms

5 Find the last term of the sequence 5, 10, 15, to 42 terms

6 Find the 18th term of the sequence $8, 3, -2, -7,$

7 Find the last term of the sequence a $a+4$, $a+8$, $a+12$, to 12 terms

8 Find the 8th term of the sequence $x+5$, $x+3$, $x+1$,

9 If $a = 5$ $l = 395$ and $d = 5$, find the value of n

10 Find the first term of an arithmetic progression when $d = 4$, $l = 73$, and $n = 24$

11 Find the common difference of an arithmetic progression when the seventh term is 17, the first term is 5, and the number of terms is 7

12 The 18th term of an arithmetic progression is -71 and the common difference is -4 Find the first term

13 Write an equation showing that m n, and p form an arithmetic progression

14 The 16th term of an arithmetic progression is 48 Find the first term if the common difference is 3

Example 2 The seventh term of an arithmetic progression is 22 and the tenth term is 31 Find the first term and the common difference

Solution $\quad a + (n-1)d = l$

Then $\quad a + 6d = 22$

and $\quad a + 9d = 31$

Subtracting $\quad -3d = -9$

$$d = 3$$

Substituting 3 for d in (1) $\quad a + 18 = 22$

$$a = 4$$

The first term is 4 and the common difference is 3

EXERCISES [A]

1 The ninth term of an arithmetic progression is 21 and the fifteenth term is 33 Find the first term, the common difference, and the first four terms of the progression

2 The fifteenth term of an arithmetic progression is 91 and the twentieth term is 121 Write the first five terms of the progression

3 The third term of an arithmetic progression is 1 and the seventh term is -11 Write the first four terms of the progression

4 The eighteenth term of an arithmetic progression is 47 and the second term is -1 Find the common difference Which term of this progression is 32?

5 Solve $l = a + (n-1)d$ for a, for n, for d

Arithmetic Means [A]

The terms which are between any two nonconsecutive terms of an arithmetic progression are called the arithmetic means between the two terms Thus in the sequence 4, 6, 8, 10, the numbers 6 and 8 are the arithmetic means between 4 and 10

The arithmetic mean between 7 and 11 in the sequence 5, 7, 9, 11, is 9 Do you see that 9 is the average of 7 and 11? When there is only one arithmetic mean between two numbers, it is the average of these numbers

Example Insert five arithmetic means between -3 and 21

Algebraic Solution $n = 7$, $l = 21$, and $a = -3$

$$l = a + (n-1)d$$
$$21 = -3 + 6d \text{ or } d = 4$$

Then the sequence is -3 1, 5, 9, 13 17, 21,

Arithmetic Solution There are 6 differences between 21 and -3 and the sum of these differences is 24 Then each difference is $\frac{1}{6}$ of 24 or 4 The sequence is -3, 1, 5, 9, 13, 17, 21,

EXERCISES [A]

1 Insert one arithmetic mean between 1 and 29
2 Insert one arithmetic mean between 7 and 109
3 Find the average of 111 and 125
4 Insert three arithmetic means between 6 and 26
5 Insert three arithmetic means between 11 and 23
6 Find the arithmetic mean between -7 and 41
7 Insert six arithmetic means between 4 and 53
8 Insert five arithmetic means between 55 and -8
9 Insert five arithmetic means between $2x - 7$ and $2x + 11$
10 Insert four arithmetic means between $a - 9$ and $a + 6$

Finding the Sum of n Terms of an Arithmetic Progression [A]

One method of finding the sum of a certain number of terms of an arithmetic progression is to add the terms but this method is too long Let us develop a formula for finding the sum of n terms

Let S = the sum of n terms of an arithmetic progression

Then

$$S = a + (a+d) + (a+2d) + \quad + (l-2d) + (l-d) + l$$

Also $S = l + (l-d) + (l-2d) + \quad + (a+2d) + (a+d) + a$

Adding these two equations we get

$$2S = (a+l) + (a+l) + (a+l) + \quad \text{to } n \text{ terms}$$

Then $2S = n(a+l)$

$$S = \frac{n}{2}(a+l)$$

$\frac{a+l}{2}$ is the average of the terms

The formula for the sum of n terms in an arithmetic progression is

$$S = \frac{n}{2}(a + l)$$

Suppose that we wish to find the sum of the first ten terms of the sequence 6 8 10 We can find l by the formula $l = a + (n - 1)d$

We find $l = 24$ By the formula $S = \frac{n}{2}(a + l)$ $S = \frac{10}{2}(6 + 24) = 150$

We shall develop a formula for finding S when n a and d are given

$$S = \frac{n}{2}(a + l) \text{ and } l = a + (n - 1)d$$

Then

$$S = \frac{n}{2}[a + a + (n - 1)d]$$

$$S = \frac{n}{2}[2a + (n - 1)d]$$

We shall now solve the problem above by this formula

$$S = \frac{n}{2}[2a + (n - 1)d]$$

$$S = \frac{10}{2}[12 + (9)2]$$

$$S = 150$$

(A) **EXERCISES**

1 Find the sum of the integers 1 to 50 inclusive

2 Find the sum of the odd integers 1 to 99 inclusive

3 Find the sum of the even integers 2 to 100 inclusive

4 Find the sum of the first eighteen terms of the sequence 3 6 9 12

5 What is the sum of the first twenty terms of the sequence 8 6 4 ?

6 Find the sum of the first twelve terms of the sequence 5 7 9

7 Find the sum of the first fifteen terms of the sequence 48 45 42

8 Find the sum of the first twenty five terms of the sequence 0 -6 -12

9 Find the sum of all the even integers between 80 and 196

10 Find the sum of the first hundred terms of the sequence 88 80 72

PROBLEMS

[A]

1 If you receive 10 cents on the first day of the year, 12 cents on the second day, 14 cents on the third day, and so on, how much will you receive on the 365th day? What is the sum of these 365 gifts?

2 Bill Blue made deposits in his school bank as follows 10 cents the first week, 13 cents the second week, 16 cents the third week, and so on until he had made 52 deposits What was the amount of his last deposit, and what was the sum of his deposits?

3 An iron ball will fall 16 1 feet the first second, 48 3 feet the second second, 80 5 feet the third second, etc How far will it fall the thirtieth second? How far will it fall in thirty seconds?

4 Mr Brown's first monthly payment on a loan was $47 15, his second was $45 93, and his third was $44 71 If his payments decreased at the same rate, how much did he pay on the loan in 1 year?

5 In a potato race 20 potatoes are placed in a straight line 5 feet apart, the first potato being placed 10 feet from the starting point How many feet does a contestant travel in collecting the potatoes one at a time and placing them in a basket stationed at the starting point?

6 If your father had given you $1 the day you were one year old, $2 the day you were two years old, $3 the day you were three years old, and continued this plan till you were fourteen years old, what would the sum of these gifts be?

7 A ball rolls down an inclined plane 5 21 feet the first second, and during each succeeding second 10 42 feet more than during the preceding second How far will it roll in 8 seconds?

8 The average of three numbers is 56 The difference between the second and first equals the difference between the third and second Find the three numbers if the sum of the first and second is 96

9. Find the sum of the first n integers

10 Derive a formula for finding d when S, a, and l are given

11. How many integers between 90 and 150 can be exactly divided by 7?

Geometric Progression [A]

A geometric progression is a sequence of numbers any one of which, after the first, can be obtained by multiplying the preceding one by the same number. For example, 4, 12, 36, 108, and 25, -5, 1, $-\frac{1}{5}$, are geometric progressions.

The common ratio of a geometric progression is the number by which any term of a geometric progression must be multiplied to equal the next term. It is the ratio of any term of the progression to the preceding term. The common ratio of the sequence 4, 12, 36, is 3, and the common ratio of the sequence 25, -5, 1, $-\frac{1}{5}$, is $-\frac{1}{5}$.

[A] **EXERCISES**

Find the fifth term and the sum of the first 5 terms of each of the following geometric sequences.

1. 2, 4, 8, 16,	5. 20, 10, 5, $2\frac{1}{2}$,
2. 1, 3, 9, 27,	6. 8, 2, $\frac{1}{2}$, $\frac{1}{8}$,
3. 5, 10, 20, 40,	7. 27, -9, 3, -1,
4. 8, -4, 2, -1,	8. 1, 0.1, 0.01, 0.001,

Formula for the Last Term of a Geometric Progression [A]

We shall now develop a formula for finding the last term in a geometric progression. Any geometric sequence is of the form a, ar, ar^2, ar^3, ar^4, In this series a is the first term and r is the common ratio.

The first term is a
the second term is ar
the third term is ar^2
the fourth term is ar^3
etc.

The exponent of r in any term is 1 less than the number of the term. Then l, the last term or the nth term, is ar^{n-1}.

In a geometric progression,

$$l = ar^{n-1}$$

Example 1. Find the eighth term of the sequence 6, 12, 24,

Solution

$$l = ar^{n-1}$$
$$l = 6(2)^7 = 6 \times 128 = 768$$

EXERCISES

[A]

Find the designated term in each of the following sequences

1. The seventh term of the sequence 1, 3, 9,
2. The fifth term of the sequence 1, -2, 4,
3. The sixth term of the sequence -8, -4, -2,
4. The fifth term of the sequence -8, $+4$, -2,
5. The seventh term of the sequence 64, -16, 4,
6. The fifth term in the sequence x, $2x^2$, $4x^3$,
7. The sixth term in the sequence $6a$, 2, $\frac{2}{3}a^{-1}$,
8. The sixth term in the sequence 3, 0.3, 0.03,
9. Use logarithms to find the twentieth term of the sequence 2, 2.50, 3.125,
10. Use logarithms to find the thirtieth term of a geometric sequence whose first term is 24 and common ratio is 1.1

Example 2 The eighth term of a geometric progression is 640 and the sixth term is 160 Find the common ratio and the first eight terms

Solution	$ar^{n-1} = l$	
For the eighth term	$ar^7 = 640$	(1)
For the sixth term,	$ar^5 = 160$	(2)
Dividing (1) by (2)	$r^2 = 4$	
	$r = \pm 2$ the common ratio	

Substituting $r = \pm 2$ in (2), we get

$$32a = 160 \quad \text{and} \quad -32a = 160$$

Then $\qquad a = 5 \quad \text{and} \quad a = -5$

One sequence is 5, 10, 20, 40, 80, 160, 320, 640,

The other is -5, 10, -20, 40, -80, 160, -320, 640

EXERCISES

[A]

Find the ratio and the first three terms of the geometric sequence in which

1. The fifth term is 48 and the seventh term is 192
2. The eighth term is 10,935 and the sixth term is 1215
3. The sixth term is 192 and the eighth term is 768
4. The fourth term is $7\frac{1}{2}$ and the sixth term is $1\frac{7}{8}$
5. The sixth term is -10 and the eighth term is $-2\frac{1}{2}$

6 The fifth term is 1075 2 and the ninth term is 275,251 2

7 Write an equation showing that x, y, and z form a geometric progression

HINT $y = xr$ and $z = yr$ Eliminate r

Geometric Means [A]

The terms which are between any two nonconsecutive terms of geometric progression are called the geometric means between those terms For example, in the sequence 3, 9, 27, 81, the terms 9 and 27 are the geometric means between 3 and 81

The geometric mean of two numbers is the one and only term which is between two nonconsecutive terms of a geometric progression The geometric mean of two numbers is the mean proportional between them The geometric mean of 1 and 25 is ± 5, and the geometric mean of 5 and 125 is ± 25

Example Insert two geometric means between 3 and 24

Solution $a = 3$, $l = 24$, and $n = 4$

$$ar^{n-1} = l$$

Then
$$3r^3 = 24$$
$$r^3 = 8$$
$$r = 2$$

The required means are 6 and 12

EXERCISES [A]

1 Insert a geometric mean between 5 and 45 Give two solutions

2 Find the mean proportional between 9 and 16 Write the first 5 terms of the geometric progression having 9 for its first term and 16 for its third term

3 Insert two geometric means between 4 and 108

4 Insert three geometric means between -2 and -32

5 Insert four geometric means between -3 and 96

6 Insert two geometric means between 4 and 62 5

7 The sixth term of a geometric progression is 96 and the ninth term is 168 Find the seventh and eighth terms

8 Find the mean proportional between 10 and $\frac{1}{10}$

9 Find the geometric mean between $1\frac{4}{5}$ and $\frac{1}{5}$

10 Find the geometric mean between $5x$ and $\frac{1}{5}x$

11 Insert two real geometric means between

a 2 and $4\sqrt{2}$ c m and n e $r\sqrt{s}$ and r^4s^2

b x^4 and x^6 d $\sqrt{21}$ and $9\sqrt{7}$ f $\frac{7}{2}\sqrt{2}$ and $\frac{7}{4}$

12 Solve $l = ar^{n-1}$ for a, for r, for n

Finding the Sum of n Terms of a Geometric Progression [A]

A formula for finding the sum of n terms of a geometric progression will now be developed

Let S denote the sum of the n terms, r denote the common ratio, and a denote the first term of a geometric progression

$$\begin{array}{ll}
\text{Then} & S = a + ar + ar^2 + \cdots + ar^{n-1} \\
M_r & rS = \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\
\hline
\text{Subtracting} & S - Sr = a - ar^n \\
& (1 - r)S = a(1 - r^n) \\
& S = \dfrac{a(1 - r^n)}{1 - r}, \text{ or } \dfrac{a(r^n - 1)}{r - 1}, \text{ or } \dfrac{ar^n - a}{r - 1}
\end{array}$$

Since $l = ar^{n-1}$, then $lr = ar^n$ Substituting lr for ar^n in the formula $S = \frac{ar^n - a}{r - 1}$, we get

$$S = \frac{lr - a}{r - 1}$$

We now have these two formulas for finding the sum of n terms in a geometric progression

$$S = \frac{ar^n - a}{r - 1} \qquad S = \frac{lr - a}{r - 1}$$

Each of these formulas contains four literal numbers Any one of them can be found when the values of the other three are known

EXERCISES [A]

Find the sum of

1 The first five terms of the sequence 10, 20, 40,

2 The first six terms of the sequence 5, 15, 45,

3 The first seven terms of the progression 2, -4, 8

4 The first five terms of the progression 5, $-1\frac{2}{3}$, $\frac{5}{9}$,

5 The first six terms of the sequence -81, 27, -9,

6 The first five terms of the sequence 360 -180 90

7 The first ten terms of the sequence $\frac{5}{16}$ $\frac{1}{4}$ $\frac{1}{5}$

8 The first four terms of the sequence -192 48 -12

9 The first eight terms of the sequence $\sqrt{8}$ $\sqrt{2}$ $\frac{1}{2}\sqrt{2}$

10 The first seven terms of the sequence $\frac{1}{3}\sqrt{3}$ 1 $\sqrt{3}$

In Exercises 11 18 the values of three of the letters a r n l and S of geometric progressions will be given Find the values of the two letters that are not given

11 $a=5$ $r=3$ $n=4$	15 $a=25$ $r=\frac{1}{5}$ $n=7$
12 $r=2$ $n=6$ $l=64$	16 $a=3$ $r=3$ $l=729$
13 $a=8$ $r=\frac{1}{2}$ $n=8$	17 $r=2$ $n=7$ $l=128$
14 $a=6$ $n=5$ $S=186$	18 $r=6$ $l=1296$ $S=1555$

19 Indicate the eighth term in the sequence

$$1 \quad (a+b)^{\frac{1}{2}} \quad (a+b)$$

20 Solve $S=\dfrac{ar^n-a}{r-1}$ for a

PROBLEMS

[A]

1 The present population of a city is 30 000 If its population should double every 10 years what would it be at the end of 80 years?

2 A golf ball is dropped from a height of 30 feet to the pavement and the rebound is one fourth the distance it drops If after each descent it continues to rebound one fourth the distance dropped what is the distance the ball has traveled when it reaches the pavement on its tenth descent?

3 If I deposit 2 cents in a savings association on January 1 6 cents on February 1 18 cents on March 1 etc throughout the year how much will I deposit on December 1? How much will I have deposited during the year?

4 You have 2 parents 4 grandparents 8 great grandparents and so on How many ancestors do you have in the last 10 generations if there are no duplications?

5 A gardener plants a lily bulb and at the end of the year has 4 new bulbs He plants the new bulbs the next year they

divide like the first one, and he has 16 bulbs If they continue dividing in this manner, how many bulbs will he have at the end of 10 years?

6 A bell jar containing 400 cubic inches is connected with an air pump At each stroke of the pump $\frac{1}{20}$ of the air is removed from the bell jar What part of the air is left after 8 strokes of the pump?

7 A farmer, wishing to keep help on his farm agreed to pay a laborer \$4 the first month, \$8 the second month, \$16 the third month, and so on for 8 months At this rate how much did the farmer pay the laborer for eight months' service?

8 Find the cost of nailing 4 shoes on a horse, each shoe requiring 6 nails if the blacksmith charges 1 cent for driving the first nail 2 cents for the second nail, 4 cents for the third nail, and so on for the remaining nails

9 A man saves \$150 the first year Each year after the first he saves 10% more than in the preceding year How long will it take him to save \$696 15?

EXERCISES
Miscellaneous

[A]

1 Find the sixth term of the sequence 1, 6, 11,

2 Find the sixth term of the sequence 2, 4, 8,

3 Find the fourth term of the sequence 54 -18, 6,

4 Find the tenth term of the sequence 1, 8, 15,

5 Insert three geometric means between $\frac{1}{8}$ and 32

6 Insert three arithmetic means between 22 and 46

7 What is the number which added to each of the numbers 1, 10, and 46 produces a geometric progression?

8 The arithmetic mean of two numbers is 50 and their geometric mean is 14 Find the numbers

9 If a clock strikes the hours and twice for each half hour, how many strokes does it make in a day?

10 If a rubber ball rebounds $\frac{1}{3}$ of the distance it falls each time and falls the first time from a height of 90 feet, how high will it rise after the sixth descent?

Infinite Geometric Series [A]

As you learned on page 369, a series is the indicated sum of the terms of a sequence. If a series has a fixed number of terms, it is a <u>finite series</u>. You have learned to find the last term and the sum of a finite number of terms. For example, you can find the sixth term and the sum of the first 6 terms of the series $1 + \frac{4}{3} + \frac{16}{9} + \cdots$.

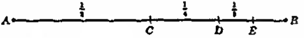
If a series has an unlimited number of terms, it is an <u>infinite series</u>. Consider the series $3 + 6 + 12 + \cdots$. In this series the terms become larger and larger. There is no last term, and we cannot find the sum of an infinite number of its terms.

Each term of the series $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \cdots$ is smaller than the preceding term, and as the number of terms of the series becomes exceedingly large, the last term becomes almost equal to zero. We say that the last term approaches zero as the limit as the number of terms is indefinitely increased. We can find the sum of this series.

Sum of an Infinite Number of Terms [A]

Let us find the sum of an infinite number of terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Do you think that the sum is as much as 50?

Let us study the problem by the use of a diagram. Let the line segment AB denote 1 foot. Suppose that a point starts at A and moves



along AB and goes $\frac{1}{2}$ foot the first second. The point at the end of the second is at C. It goes $\frac{1}{2}$ the remaining distance, CB, the next second, and reaches the point D. The third second it goes $\frac{1}{2}$ the distance DB and reaches E. If the point continues this procedure, the point will almost reach B. We say that the point approaches B as its limit. The limiting value of the total distance the point moves is the limiting value of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which is 1. We say that the sum of this series is 1.

Let us consider any geometric series of which the ratio has an absolute value less than 1. Examples of this kind of ratio are $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{4}{5}$, and .85. As n increases without limit, the nth term decreases and approaches zero as its limit.

For any finite value of n, the sum of the first n terms is given by the formula $S = \frac{ar^n - a}{r - 1}$. As n is increased indefinitely, r^n approaches

zero as its limit Then ar^n approaches zero as its limit and $\frac{ar^n - a}{r - 1}$ approaches $\frac{-a}{r-1}$ or $\frac{a}{1-r}$ as its limit We say that $\frac{a}{1-r}$ is the sum of the series

The sum of an infinite number of terms of a geometric progression in which the ratio is numerically less than 1 is given by the formula

$$S = \frac{a}{1-r}$$

Example 1 Find the sum to infinity of the series $6 + 2 + \frac{2}{3} + \frac{2}{9} +$

Solution $a = 6$ and $r = \frac{1}{3}$

$$S = \frac{a}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{\frac{2}{3}} = 9$$

EXERCISES

Find the sum of an infinite number of terms of

1 $12 + 3 + \frac{3}{4} +$

2 $5 + 1 + \frac{1}{5} + \frac{1}{25} +$

3 $16 + 8 + 4 +$

4 $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} +$

5 $\frac{3}{2} + 1 + \frac{2}{3} +$

6 $1 + \frac{1}{3} + \frac{1}{9} +$

7 $4 + 1\frac{1}{3} + \frac{4}{9} +$

8 $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$

9 $2 + \sqrt{2} + 1 +$

10 $1 + x + x^2 +$ when $|x| < 1$

11 $5 + 3\frac{1}{3} + 2\frac{2}{9} +$

12 $\frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 +$

Example 2 [8] Change .454545 to a proper fraction

Solution In this example $a = .45$ and $r = .01$

$$S = \frac{a}{1-r} = \frac{0.45}{1-.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{11}$$

[8]

EXERCISES

Express the following repeating decimals as common fractions

1 0.2222

2 0.636363

3 0.353535

4 0.3333

5 0.414141

6 0.3636

7 0.0909

8 0.123123

Express as mixed numbers

9 2.123123

10 1.901501501.50

11 4.7272

12 5.428571428571

PROBLEMS

[A]

1 A rubber ball is thrown vertically upward to a height of 70 feet If it rebounds $\frac{1}{2}$ the distance of the fall, how far does it go on the eighth rebound?

2 A square has an area of 16 square feet A second square is formed by joining the midpoints of the sides of the first square A third square is formed by joining the midpoints of the sides of the second square Find the sum of the areas of the squares if this process is continued indefinitely

3 A pendulum bob moves 20 inches on the first swing On each succeeding swing it moves one eighth less than it did during the previous swing How far will it travel before coming to rest?

4 A rubber ball is dropped from a height of 12 feet, rebounds one fourth of this distance, and continues to rebound after each descent one fourth the distance it fell Over what distance will it have passed if it continues indefinitely?

5. If you deposit \$10 every six months in a loan association which pays 2% interest, compounded every 6 months, how much will you have to your credit at the end of 20 years? Assume that each deposit is made at the beginning of the six-month period The first sum is $10(1.01)^{40}$

Some Applications of Progressions [C]

Compound Interest Did you notice that the right member of the compound interest formula on page 309 is the nth term of a geometric progression having its first term $P(1 + r)$ and its common ratio $1 + r$? The following paragraph shows the steps in developing the formula for the *compound amount* (principal plus compound interest) on P dollars at the end of n years if the interest at rate r is payable annually

Since the interest on P dollars for one year at rate r is Pr, the amount for the year will be $P + Pr$, or $P(1 + r)$ Since the amount for the first year becomes the principal for the second, the amount at the end of two years will be

$$P(1 + r) + r[P(1 + r)] = P(1 + r)(1 + r) = P(1 + r)^2$$

Similarly, the amount at the end of three years will be

$$P(1 + r)^2 + r[P(1 + r)^2] = P(1 + r)^2(1 + r) = P(1 + r)^3$$

Now it can be seen that the amounts form a geometric progression

$$P(1+r),\ P(1+r)^2,\ P(1+r)^3,$$

in which the common ratio is $1+r$. To find the nth term of this progression we use the formula $l = ar^{n-1}$ and obtain

$$A = P(1+r)(1+r)^{n-1}$$

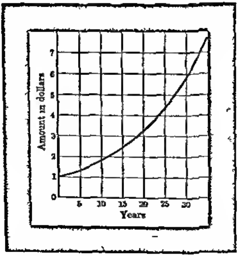
$$A = P(1+r)^n \qquad (1)$$

Thus, the compound amount on P dollars at rate r for n years is given by the formula $A = P(1+r)^n$

When the interest is compounded k times a year instead of annually, the rate for each period becomes $\frac{r}{k}$ and the number of interest periods becomes kn where n is the number of years. The formula for the amount on P dollars at rate r compounded k times per year for n years is then

$$A = P\left(1+\frac{r}{k}\right)^{kn} \qquad (2)$$

Formulas for Organic Growth and Decay. Do you recognize that compound interest represents growth? As time elapses, not only does the



Amount of $1 at 6% compounded annually

amount grow, but it grows more rapidly This is shown dramatically by the graph on page 384 which shows the compound amount of $1 at 6% compounded once per year If the interest is compounded more than once per year, the amount grows even more rapidly The increase is only slight, however, because as the number of interest intervals grows larger, the length of the intervals grows smaller, thus cutting down on the amount of interest which can accumulate during the interval Do you see that as k in formula (2) becomes larger, the compounding becomes more nearly continuous?

Since $\frac{1}{\frac{k}{r}} = \frac{r}{k}$ and since $\frac{k}{r} \cdot nr = kn$, we can rewrite formula (2) as

$$A = P\left[\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}\right]^{nr} \tag{3}$$

It can be shown by methods to be demonstrated in the next chapter that as k approaches infinity, the value of $\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}$ approaches the number 2 718 which we symbolize by e If we represent by Q the limit of A, as k approaches infinity, the formula becomes

$$Q = Pe^{nr} \tag{4}$$

We call the formula the formula for organic growth because it can, within limits, explain many natural phenomena of growth

Bacteria, under favorable conditions, multiply according to the formula $Q = Pe^{nr}$ While the formula cannot be applied exactly to the matter of compound interest, since it would be impossible to compound interest an infinite number of times per year, it does approximately fit the situation in large companies where interest is received and loans made many times each day

The formula $Q = Pe^{-nr}$ results when r in formula (1) is replaced by $-r$ to give a formula for discount We call the formula $Q = Pe^{-nr}$ the formula for decay You know that as you rise above sea level, the pressure of the air around you decreases, but do you know that the pressure at a given height can be found by the formula $Q = Pe^{-nr}$? Do you know that a pan of hot water or a motor cools in open air according to the formula? Radioactive materials disintegrate according to the same pattern Telephone companies must set up, at in-

tervals stations to step up the intensity of the voice because, on long distance telephone calls, conversation fades out at a rate which fits the formula

During World War I a group of doctors showed that if a wound heals normally, the area of the wound fits the pattern $Q = Pe^{-nr}$. In many cases where the rate of decrease of the size of the wound did not follow the pattern of the formula, the doctors were able to detect infection before it was visible. By being able to start treatment earlier, they probably saved lives that would otherwise have been lost.

Annuities. An annuity is a series of payments, usually equal in amount, made at equal intervals of time. Rent paid for a house, premiums paid on insurance, and pensions are forms of annuities. If a man deposits P dollars in a bank each year for n years and the money earns interest at the rate of r per cent, compounded annually, his annuity increases in value. To find its amount at the end of n years we reason as follows.

If the man deposits P dollars in the bank at the end of each year for n years, the first year's deposit will earn interest for $n-1$ years, the second for $n-2$ years, the third for $n-3$ years, and so on, with the last deposit earning no interest (interest for $n-n$ years). At the end of n years

The amount of the first deposit will be $P(1+r)^{n-1}$
The amount of the second deposit will be $P(1+r)^{n-2}$
The amount of the third deposit will be $P(1+r)^{n-3}$

The amount of the $(n-1)$st deposit will be $P(1+r)^{n-n+1}$
The amount of the nth deposit will be $P(1+r)^0$

The sum of these amounts will be amount of the annuity at the end of n years. If we add them in reverse order, that is, with the last amount first, and so on, we have

$$P + P(1+r) + \cdots + P(1+r)^{n-3} + P(1+r)^{n-2} + P(1+r)^{n-1}$$

We recognize this as a geometric series of n terms having a common ratio of $1+r$. The sum of this series is

$$A = \frac{(1+r)\cdot P(1+r)^{n-1} - P}{1+r-1}$$

$$A = P\left[\frac{(1+r)^n - 1}{r}\right] \tag{5}$$

Formula (5) is then the formula for the amount of an annuity of P dollars per year if the annuity is invested at rate r compounded annually

When the interest is compounded k times per year, we may consider that rate per interval to be $\frac{r}{k}$ and the number of intervals during the n years as kn Accordingly the formula becomes

$$A = P\left[\frac{\left(1+\frac{r}{k}\right)^{kn}-1}{\frac{r}{k}}\right] \tag{6}$$

Checking Your Understanding of Chapter 13

At this point you should check your understanding of the principles mentioned in Chapter 13 and your ability to handle the operations described in it Make sure that you can

Check yourself

	PAGE
1 *Recognize an arithmetic progression*	369
2 Find the last term of an arithmetic progression	369
3 Form an arithmetic progression by inserting a desired number of arithmetic means between two given numbers	371
4 *Find the sum of the terms of a finite arithmetic* progression	372
5 Recognize a geometric progression	375
6 Find the last term of a geometric progression	375
7 Form a geometric progression by inserting a desired number of *geometric means* between *two* given numbers	377
8 Find the sum of the terms of a finite geometric progression (p 378) and of an infinite geometric series having the absolute value of its ratio less than 1	381

9 Give the *meaning of, and* spell correctly, the *expressions at the top of the next page*

MATHEMATICAL VOCABULARY

	PAGE
arithmetic means	371
arithmetic progression	369
common difference	369
common ratio	375
finite series	381
geometric means	377
geometric progression	375
infinite series	381
mean proportional	377
progression	369
sequence	369
series	369

CHAPTER REVIEW

(A)

1 How is each term of an arithmetic progression obtained from the preceding term?

2 How is each term of a geometric progression obtained from the preceding term?

3 What is the ratio in the sequence $6, -3, 1\frac{1}{2}, \cdots$? in the sequence 14, 35, 87.5, $\cdots$?

4 What is the common difference in the progression $-4, -1, 2, \cdots$?

5 Write the formulas for

a The last term of an arithmetic progression

b The last term of a geometric progression

c The sum of n terms of an arithmetic progression

d The sum of n terms of a geometric progression

e The sum of an infinite number of terms of a geometric progression which has a ratio with absolute value less than unity

6 $y-x=z-y$. Do x, y, and z form an arithmetic progression or a geometric progression?

7 Three consecutive terms of a geometric progression are a, b, and c. Write an equation showing the relation of a, b, and c.

8 Find the 24th term of an arithmetic progression whose first term is 50 and common difference 5

9 Find the 7th term of a geometric sequence whose first term is 49 and common ratio 2

10 Find the sum of the first 8 terms of the sequence 36, 12, 4,

11 Find the common ratio of the sequence 3^0, $3^{\frac{1}{3}}$, $3^{\frac{2}{3}}$, 3,

12 Find the difference between the geometric mean of 9 and 25 and the arithmetic mean of 9 and 25

13 Insert 4 arithmetic means between 12 and 56

14 Insert 5 geometric means between $2\frac{1}{4}$ and $\frac{16}{81}$

15 Find the fifth term of the sequence $2+\sqrt{2}$, $2-\sqrt{2}$, $2-3\sqrt{2}$,

16 Find the fifth term of the sequence $1+\sqrt{3}$, $\sqrt{3}+3$, $3+3\sqrt{3}$,

17 Insert 3 real geometric means between 2 and 50

18 Find the 12th term of the sequence 1.03, 1.08, 1.13,

19 Find the 12th term of the sequence 1.03, 1.03^2, 1.03^3,

20 If each year an automobile depreciates 30% of its value the preceding year, find the value of a car which cost $1980 four years ago

CHAPTER TESTS

[Test A]

1 Find the fourth term of each sequence

a $\frac{5}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$ b $\frac{5}{4}$, 1, $\frac{4}{5}$

2 Write formulas for

a The nth term of an arithmetic progression

b The nth term of a geometric progression

c The sum of n terms of an arithmetic progression

d The sum of n terms of a geometric progression

3 Find the twentieth payment if the first three payments are \$4 50, \$4 35, \$4 20

4 Find the sum of the payments in exercise 3

5 Find the sum of the first 10 terms of the sequence 6, 12, 24,

6 Find the eighth term of the sequence 16, 32, 64,

7 Find the seventh term of the sequence 1 04, $1\ 04^2$, $1\ 04^3$,

8 Find the sum of the odd numbers from 7 to 151 inclusive

9 Find the sum of the first 6 terms of the sequence 432, -144, 48,

10 Insert three geometric means between 3 and $\frac{1}{27}$

11 Find the first term of an arithmetic progression of 10 terms if the common difference is 7 and the last term is 108

12 A clock strikes once on each quarter hour, twice on the half hour, three times on the three quarter hour, and at the end of each hour strikes the hour How many times will it strike in 12 hours?

CHAPTER 14

The Binomial Theorem

In this chapter we learn a short method of finding powers of binomials

The Binomial Formula [A]

The binomial theorem is a formula for expanding any integral power of a binomial without performing the actual multiplication

If we perform the multiplication indicated by the integral powers of the binomial $(a+b)$, we have

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a + b\\
(a+b)^2 &= a^2 + 2ab + b^2\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{aligned}
$$

If we remove the literal terms from the above expansion, we have an arithmetic triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

This triangle, known as Pascal's triangle, was named for the French mathematician Blaise Pascal (1623–1662) The triangle is bordered by 1's on two sides The numbers in each row give the coefficients of the expansion of a binomial Have you discovered how the numbers in any horizontal row are found? The first and last number in each row is 1 To find each of the other numbers in the row, add each of the two numbers nearest it in the row above For example, the numbers in the sixth row are obtained in this manner

$$
\begin{array}{cccccc}
0+1=1 & 1+4=5 & 4+6=10 & 6+4=10 & 4+1=5 & 1+0=1\\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
$$

Pascal's triangle may be used to find the coefficients of the expansion of $(a+b)$ to any integral power

If we let n represent the exponent of $(a+b)$ in any of the expressions of $(a+b)$ and make a study of the products, we observe that

1 The first term is a^n and the last term is b^n

2 The second term is $na^{n-1}b$

3 The exponents of a decrease by 1 and the exponents of b increase by 1 in each successive term

4 *If in any term the coefficient is multiplied by the exponent of a,* and divided by the number of that term, the result is the coefficient of the next term

5 The number of terms is $n + 1$

From these relations we can write a formula for $(a + b)^n$ when $n < 7$ It is as follows

Binomial Formula

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1} + \frac{n(n-1)a^{n-2}b^2}{1 \cdot 2} + \frac{n(n-1)(n-2)a^{n-3}b^3}{1 \cdot 2 \cdot 3}$$
$$+ \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots + b^n$$

In this chapter we shall assume that the formula is true for all values of a and b and for all positive integral values of n Later (page 551) we shall prove the formula true for all positive integral values of n

The Factorial Symbol[A]

In expanding binomials and in higher mathematics it is often desirable to indicate certain products by symbols For example, we indicate the product $1 \times 2 \times 3 \times 4$ by $4!$ Likewise we indicate the product $1 \times 2 \times 3 \times 4 \times 5 \times 6$ by $6!$, which is read "factorial six" The symbol $n!$ (read *factorial n*) indicates the product of all the positive integers from 1 to n inclusive

Since $n! = n(n-1)!$ is true for all positive integral values of n, it is necessary to define $0!$ We define $0!$ as equal to 1

Example Find the value of $7!$

Solution $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$

EXERCISES [A]

Find the values of the following

1 $\frac{6!}{3!}$ 3 $\frac{9!}{8!}$ 5 $\frac{8!}{4!2!}$ 7 $\frac{n!}{(n-1)!}$

2 $\frac{5!}{2!}$ 4 $\frac{3!3!}{6!}$ 6 $\frac{2!4!}{6!}$ 8 $\frac{2\,n!}{n!}$

The factorial notation may be used in writing the binomial formula, as follows

$$(a+b)^n = \frac{a^n}{0!} + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!} + \quad + b^n$$

Example 1 Expand $(x+y)^4$ by the binomial formula

Solution. $(x+y)^4 = x^4 + \frac{4x^3y}{1!} + \frac{12x^2y^2}{2!} + \frac{24xy^3}{3!} + \frac{24y^4}{4!}$
$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Example 2 Expand $(a-b)^5$ by the binomial formula

Solution. $(a-b)^5 = a^5 + \frac{5a^4(-b)}{1!} + \frac{20a^3(-b)^2}{2!} + \frac{60a^2(-b)^3}{3!} + \frac{120a(-b)^4}{4!} + \frac{120(-b)^5}{5!}$
$= a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

(A) **EXERCISES**

Expand by the binomial formula

1 $(a+b)^7$ 3 $(c-d)^7$ 5 $(a-b)^8$
2. $(x-y)^6$ 4 $(x+y)^8$ 6. $(c+d)^9$

Write only the first four terms of

7. $(a+b)^{12}$ 8. $(x-y)^{16}$ 9 $(c+d)^{20}$

Example 3 Expand $(2x-3y)^5$

Solution.

$(2x-3y)^5 = (2x)^5 + \frac{5(2x)^4(-3y)}{1!} + \frac{20(2x)^3(-3y)^2}{2!} + \frac{60(2x)^2(-3y)^3}{3!} + \frac{120(2x)(-3y)^4}{4!} + \frac{120(-3y)^5}{5!}$
$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

Example 4 Expand $(3x^3-1)^4$

Solution.

$(3x^3-1)^4 = (3x^3)^4 + \frac{4(3x^3)^3(-1)}{1!} + \frac{12(3x^3)^2(-1)^2}{2!} + \frac{24(3x^3)(-1)^3}{3!} + \frac{24(-1)^4}{4!}$
$= 81x^{12} - 108x^9 + 54x^6 - 12x^3 + 1$

EXERCISES [A]

Expand

1. $(a-2)^5$ 5. $(5x+1)^4$ 9. $(1-x)^5$ 13. $(2x-\frac{1}{2}y)^4$
2. $(a+2b)^6$ 6. $(2x-1)^6$ 10. $(a+2)^7$ 14. $(y+\frac{1}{2})^4$
3. $(2x-y)^4$ 7. $(3-2c)^5$ 11. $(2-x)^6$ 15. $(x^3+1)^5$
4. $(1+x)^7$ 8. $(1-5m)^4$ 12. $(a^2-4)^4$ 16. $(3x+2y)^3$

17. $(a^{\frac{1}{2}}+2b^2)^4$ 19. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^4$ 21. $(m^{-1}+n^{-1})^3$

18. $(a-b^{\frac{1}{2}})^5$ 20. $\left(\frac{x}{2}-\frac{y}{3}\right)^4$ 22. $(x^4+1)^5$

Example 5 [B] Find the value of $(99)^3$

Solution. $99^3=(100-1)^3$

$$(100-1)^3=(100)^3+\frac{3(100)^2(-1)}{1!}+\frac{6(100)(-1)^2}{2!}+\frac{6(-1)^3}{3!}$$
$$=1{,}000{,}000-30{,}000+300-1=970{,}299$$

Example 6 [B] Find the value of $(1.04)^8$ to the nearest thousandth

Solution $(1.04)^8=(1+.04)^8$

$$(1+.04)^8=1^8+\frac{8\cdot1^7(.04)}{1!}+\frac{56\cdot1^6(.04)^2}{2!}+\frac{336\cdot1^5(.04)^3}{3!}$$
$$+\frac{1680\cdot1^4(.04)^4}{4!}+\cdots$$
$$=1+.32+.0448+.003584+.000179+\cdots$$
$$=1.369$$

The series was terminated when the value of the term became less than .0001. Do you see how the binomial formula can be used to find compound amounts?

EXERCISES [B]

Find the value of

1. $(98)^3$ 2. $(19)^4$ 3. $(102)^5$ 4. $(31)^4$
5. Find the value of $(1.02)^8$ to the nearest hundredth
6. Find the value of $(0.8)^{10}$ to the nearest hundredth
7. Find the value of $(1.2)^9$ to the nearest thousandth
8. Find the value of $(1.7)^5$ to the nearest thousandth

Finding Any Term of an Expansion [A]

Sometimes it is necessary to write a certain term of the binomial expansion, or to find a term that has a certain exponent for one of

the letters Let us study the various terms in the general expansion as given on page 394

The 2d term is $\frac{na^{n-1}b}{1!}$

The 3d term is $\frac{n(n-1)a^{n-2}b^2}{2!}$

The 4th term is $\frac{n(n-1)(n-2)a^{n-3}b^3}{3!}$

The 5th term is $\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!}$

By examining these terms, we observe that

1 The exponent of b is one less than the number of t

2 The exponent of a is n minus the exponent of b for that term

3 The number of factors in the numerator and also in the denominator of the coefficient is the same as the exponent of b for that term

If we let r represent the number of the term we wish to find, then the

$$r\text{th term} = \frac{n(n-1)(n-2) \quad (n-r+2)a^{n-r+1}b^{r-1}}{(r-1)!}$$

Example 1 Find the 5th term of $(x-y)^{10}$

Solution From the formula for the rth term, we have,

$$5\text{th term} = \frac{10(10-1)(10-2)(10-3)}{4!}x^{10-5+1}y^{5-1}$$

$$= \frac{10}{1}\ \frac{9}{2}\ \frac{8}{3}\ \frac{7}{4}x^6y^4 = 210\,x^6y^4$$

Note the importance of the number 4, that is, $5-1$, in finding the fifth term There are 4 numerical factors in the numerator *Factorial 4 is in the denominator The exponent of* y is 4, and the exponent of x is $10-4$

Example 2 Find the term in the expansion of $(1-x^2)^8$ that contains x^{10}

Solution From the formula for the rth term, we know that the exponent of x in the required term is $(x^2)^{r-1}$ Since $(x^2)^{r-1}$ is x^{2r-2}, $2r-2=10$ or $r=6$ Hence the 6th term of the expansion will contain x^{10}

$$\text{Then the 6th term} = \frac{8(8-1)(8-2)(8-3)(8-4)1^{8-6+1}(-x^2)^{6-1}}{5!}$$

$$= -\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4\, x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$= -56\, x^{10}$$

Example 3 Find the middle term of $(3x + y)^6$.

Solution The middle term is the 4th term.

$$\text{The 4th term} = \frac{6(6-1)(6-2)}{1 \cdot 2 \cdot 3}(3x)^3y^3$$

$$= 540\, x^3y^3$$

EXERCISES

[A]

Find the indicated term in the expansion of each of the following

1. The 4th term of $(x + y)^{17}$
2. The 6th term of $(a - b)^{14}$
3. The 7th term of $(a + b)^{10}$
4. The 10th term of $(2x - y)^{13}$
5. The 15th term of $(x - y)^{20}$
6. The 7th term of $(2a + 3b)^7$
7. The 8th term of $(x - y)^{12}$
8. The 7th term of $(x^{\frac{1}{2}} - 3)^{10}$
9. The middle term of $(x + 3y)^8$
10. The middle term of $(2x - y)^{10}$
11. The middle terms of $(a - 2b)^7$
12. The middle terms of $(a^2 - 1)^9$
13. The term containing x^8 in $(x + y)^{15}$
14. The term containing x^{10} in $(x^2 + 1)^{10}$
15. The term containing y^7 in $(1 - 2y)^{12}$
16. The term containing x^6 in $(1 - 3x^2)^{10}$

even for mor

The Binomial Series [B]

If we expand $(1 + x)^n$ by the binomial formula, we have what is known as the binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+2)x^{r-1}}{(r-1)!} + \cdots$$

The right member is called the <u>binomial series</u>. It is valid for all values of x when n is a positive integer or zero.

When n is integral, the binomial series terminates with x^n When n is fractional, the series has an infinite number of terms It is proved in advanced mathematics that this theorem is valid when n is fractional if $-1 < x < 1$

Any binomial may be written in the form of the binomial series Thus $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$ where $-1 < \frac{b}{a} < 1$

Example Find $\sqrt[3]{26}$ to the nearest hundredth

Solution $\sqrt[3]{26} = 26^{\frac{1}{3}} = (27-1)^{\frac{1}{3}} = 3(1-\frac{1}{27})^{\frac{1}{3}}$

$$3(1-\tfrac{1}{27})^{\frac{1}{3}}$$
$$= 3\left[1 - \frac{1}{81} + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{1}{27})^2}{2!} + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{1}{27})^3}{3!} + \quad\right]$$
$$= 3[1 - 0123 - 0002 + \quad]$$
$$= 3[\,9875] = 2\ 96$$

The series was terminated with the third term because the value of the fourth term would have no effect on the accuracy of the root to the nearest hundredth

Of course finding roots by means of logarithms is much simpler The binomial series is given here because it is an important series in more advanced mathematics

(B)

EXERCISES

Find the following roots to the nearest hundredth, using the binomial series

1 $\sqrt[4]{17}$ 2 $\sqrt{24}$ 3 $\sqrt[3]{9}$ 4 $\sqrt[3]{65}$

Checking Your Understanding of Chapter 14

At this time you should check how thoroughly you have mastered this chapter Be sure that you know

Should you review

1 How to make Pascal s Triangle and how to use it when finding an integral power of a binomial (p 393)

2 How to write the binomial formula (p 394)

3 The meaning of factorial n (p 394)

4 How to expand a binomial by the binomial formula (p 395)

5 How to find any term in a binomial expansion (p 396)

6 That $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ is valid

(*a*) when n is zero or a positive integer (p 398)

(*b*) when n is a fraction provided $-1 < x < 1$ (p 399)

CHAPTER TESTS

[Test A]

1 Find the value of $8!$

2 How many terms are there in the expansion of $(a+b)^{12}$?

3 What is the difference between a series and a sequence?

4 Write the fourth row of Pascal's Triangle

5 Expand $(a-b)^8$

6 Expand $(2x+3y)^5$

7 Expand $(2x^2-1)^6$

8 Evaluate $\frac{192}{2!4!}$

9 Find the 6th term in the expansion of $(x+y)^{12}$

10 Find the term containing s^5 in the expansion of $(r+2s)^{10}$

[Test B]

1 Evaluate $\frac{(n+1)!}{(n-1)!}$

2 Expand $(5a-2b)^6$

3 Expand $(3x^2-\frac{1}{2})^5$

4 Find the middle terms of $(3r+s^2)^7$

5 Find the term containing x^6 in the expansion of $(3x^2+y)^7$

6 Find the value of $(1.2)^5$ by the binomial theorem

7 Find $\sqrt[4]{15}$ by the binomial series

CUMULATIVE TEST (Chapters 9-14)

[B]

Solve each problem and write the answer (a), (b), (c), or (d) which you think is correct Give only one answer for each problem

1 Find the maximum value of the function $-3x^2+6x-3$

a 0

b 2

c -3

d 3

2 The solution of $\frac{x}{x-1}+\frac{x}{x+1}=\frac{x+6}{x^2-1}$ is

a $x=2$, or -3

b $x=-2$, or -3

c $x=2$, or -1

d $x=2$, or $-\frac{3}{2}$

3. Form the equation whose roots are 2 and $-\frac{1}{2}$

a. $2x^2 + 3x + 2 = 0$
b. $2x^2 - 3x = 2$
c. $x^2 + x - 1 = 0$
d. Not (a), (b), or (c)

4. The solution of $\sqrt{x^2 + 7} = -4$ is

a. $x = 3$
b. $x = -3$
c. $x = \pm 3$
d. No solution

5. If $\log(x^2 - 2x + 2) = 1$, the value of x is

a. 2, or 4
b. 4, or -2
c. 2, or 10
d. 1, or -2

6. If $\log_{10} 2$ is .3010, find $\log_{10} 160$

a. 1.2040
b. 3.2040
c. 2.9030
d. 2.2040

7. $\log x + \log y = \log(x - y)$ only if

a. $x = y = 1$
b. $x = y = 0$
c. $x = \dfrac{y}{1 - y}$
d. $x = \dfrac{y}{1 + y}$

8. Solve $2^{x+1} = 64$

a. $x = 4$
b. $x = 6$
c. $x = 5$
d. Not (a), (b), or (c)

9. If $\sin 17° = .2924$ and $\sin 18° = .3090$, find without the use of a table $\sin 17° 20'$

a. .2953
b. .2961
c. .3007
d. .2979

10. The sine of $\angle x = \frac{3}{5}$. Find the value of tangent $\angle x$

a. $\frac{4}{5}$
b. $\frac{4}{3}$
c. $\frac{3}{4}$
d. $\frac{5}{3}$

11. In $\triangle ACB$, $\angle C = 90°$, $\angle A = 40°$, and $BC = 18.4$. Find the correct equation for solving for AB

a. $\dfrac{18.4}{AB} = \sin 40°$
b. $\dfrac{18.4}{AB} = \tan 40°$
c. $\dfrac{AB}{18.4} = \sin 40°$
d. $\dfrac{AB}{18.4} = \tan 40°$

12. The graph of $x^2 = 25 - 4y^2$ is

a. a circle c. a parabola

b. a pair of lines d. an ellipse

13. The solution of the pair of equations

$$x - 3y = -3$$
$$x^2 + y^2 = 13$$

is

a. $x = -3,\ 3$; $y = 2,\ -2$ c. $x = 3,\ \frac{4}{5}$; $y = 2,\ -\frac{1}{5}$

b. $x = 3,\ -\frac{18}{5}$; $y = 2,\ -\frac{1}{5}$ d. Not (a), (b), or (c)

14. An airplane made a flight of 480 miles and returned Its rate going was $\frac{3}{4}$ the rate returning, and it required 40 minutes longer to go than to return Find the rate returning

a. 200 m p h c. 240 m p h

b. 256 m p h d. 180 m p h

15. The first three terms of an arithmetic progression are $x + 1$, $4x$, and $5x + 9$ Find the fourth term

a. 20 c. 45

b. 34 d. 48

16. Change 1333 · to a proper fraction

a. $\frac{4}{3}$ c. $\frac{2}{15}$

b. $\frac{4}{13}$ d. $\frac{1}{5}$

17. Insert three geometric means between -4 and -324

a. $12, -36, 108$ c. $12, 36, 108$

b. $-12, 36, -108$ d. Not (a), (b), or (c)

18 Find the 5th term of the expansion of $(2x - 1)^8$

a. $-1798x^5$ c. $-1120x^4$

b. $1120x^4$ d. $-560x^4$

19 Which term in the expansion of $\left(a - \frac{1}{a}\right)^8$ does not contain a?

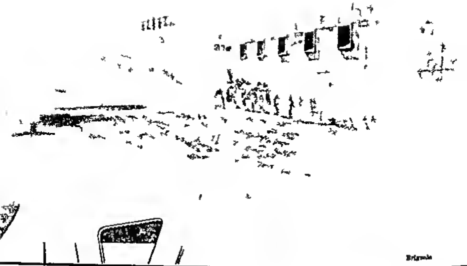
a. 2nd c 4th

b. 6th d 5th

20. Find the middle term of the expansion of $(\sqrt{3} - \sqrt{2})^5$

a. $60\sqrt{2}$ c. $-120\sqrt{6}$

b $120\sqrt{3}$ d. Not (a), (b), or (c)



Brignola

MATHEMATICS IN ARCHITECTURE

Brown Brothers

Before the construction of a large building can be started, a vast amount of study, planning, and computation must be done The architect must be a practical dreamer, seeing in his mind the outside and the inside of the building, visualizing all the activities that will take place within it, and picturing it in relation to its surroundings. He must be sure that the building is practical and beautiful, and that its construction is economical The pictures show modern buildings at the University of Florida and the Harvard Graduate School

Much mathematics is needed in designing the building The size and weight of the building, the structure of the subsoil, and the local weather conditions are factors that must be considered in determining the kind of footing to be used The various stresses must be computed and the sizes of the structural parts of the building determined

Students who contemplate entering a college of architecture direct from high school should have had at least one and a half years of algebra, one year of plane geometry, and a half year of solid geometry

In an architectural college the student may expect to study algebra, trigonometry, analytic geometry, and descriptive geometry In some colleges calculus is required

Some architectural colleges require college training for admission, selecting only those applicants who show the greatest aptitude for architectural work.

CHAPTER 15

Advanced Topics in Quadratic Equations

In this chapter you will extend your knowledge of second-degree equations

In this chapter you will learn how to determine the character of the roots of a quadratic equation without solving the equation For example, you will be able to find the sum and the product of the roots of a quadratic equation by inspection

Character of the Roots of a Quadratic Equation[A]

The solution of the general quadratic equation $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Let x_1 denote one of these roots and x_2 represent the other Then

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The radicand $b^2 - 4ac$ in the value of x is called the discriminant of the equation because it tells so much concerning the roots of the equation

The discriminant of a quadratic equation may equal zero, may be a perfect square, may be positive but not a perfect square, or may be negative We shall now study these cases in order

1 If $\underline{b^2 - 4ac = 0}$, then $x_1 = \frac{-b+0}{2a}$ and $x_2 = \frac{-b-0}{2a}$ Each root equals $\frac{-b}{2a}$ Then both roots are real, rational, and equal Thus, in the equation $x^2 - 8x + 16 = 0$, $b^2 - 4ac = 64 - 4(1)(16) = 0$, $x_1 = \frac{8+0}{2} = 4$, and $x_2 = \frac{8-0}{2} = 4$ Notice that the graph of this equation (Fig 1) intersects the x-axis in the points $x_1 = 4$ and $x_2 = 4$, which coincide The x-axis is tangent to the parabola at (4, 0)

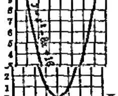


Fig 1

2 If $b^2 - 4ac$ is positive and a perfect square, neither root in its simplest form contains a radical Then both roots are real, rational, and unequal

In the equation $2x^2 - 5x - 3 = 0$, $b^2 - 4ac = 49$ The solution is

$$x = \frac{5 \pm \sqrt{25 - 4(2)(-3)}}{4} = \frac{5 \pm \sqrt{49}}{4}$$

$$= \frac{5 \pm 7}{4}$$

$$x_1 = \frac{5+7}{4} = 3,$$

$$x_2 = \frac{5-7}{4} = -\frac{1}{2}$$

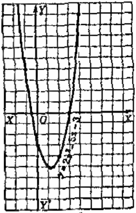


Fig 2

Since 49 is a perfect square, both roots are rational Notice that the graph (Fig 2) intersects the x-axis in the points whose abscissas are 3 and $-\frac{1}{2}$

3 If $b^2 - 4ac$ is positive but not a perfect square, the roots are real, irrational, and unequal

In the equation $x^2 + 5x + 3 = 0$,

$b^2 - 4ac = 13$,

$$x_1 = \frac{-5 + \sqrt{13}}{2},$$

$$x_2 = \frac{-5 - \sqrt{13}}{2}.$$

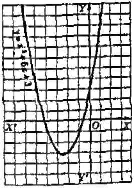


Fig 3

The graph (Fig 3) intersects the x axis in the points whose abscissas are approximately $-0\,7$ and $-4\,3$

4 If $b^2 - 4ac$ is negative, both roots are imaginary. In the equation $x^2 - x + 2 = 0$,

$$x_1 = \frac{1 + \sqrt{-7}}{2}, \quad \text{and} \quad x_2 = \frac{1 - \sqrt{-7}}{2}.$$

Notice that the graph (Fig 4) does not touch or intersect the x-axis in any real points If one root of a quadratic equation is imaginary, the other root is imaginary Why?

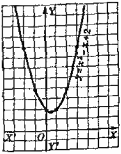


Fig 4

We shall now summarize the facts which have been presented

In the equation $ax^2 + bx + c = 0$,

1. If $b^2 - 4ac = 0$,
   the roots are real, rational, and equal
2. If $b^2 - 4ac$ is positive and a perfect square,
   the roots are real, rational, and unequal
3. If $b^2 - 4ac$ is positive but not a perfect square,
   the roots are real, irrational, and unequal
4. If $b^2 - 4ac$ is negative,
   the roots are imaginary

To help you remember the facts just given, keep in mind a mental picture of the quadratic formula

EXERCISES [A]

Use the discriminant to determine the character of the roots of each of the following equations

1. $x^2 + 5x - 3 = 0$
2. $x^2 + 9x + 14 = 0$
3. $x^2 - 5x - 4 = 0$
4. $4x^2 + 3x = 0$
5. $x^2 - 6x = -9$
6. $6y^2 + 6 = -10y$
7. $2m^2 + 3 = m$
8. $x^2 + x + \frac{1}{2} = 0$
9. $4x^2 + 1 = 4x$
10. $6y^2 + 13y + 6 = 0$
11. $x^2 - x - 1 = 0$
12. $x^2 - x + 1 = 0$

Example Find the value of k for which the equation $kx^2 + 4x - 4 = 0$ has equal roots

Solution If the equation has equal roots, the discriminant equals zero Then $4^2 - 4k(-4) = 0$ Solving, $k = -1$

EXERCISES [A]

Find the values of k for which each of the following equations will have equal roots

1. $4x^2 - 20kx + 25 = 0$
2. $9y^2 - 6y + k = 0$
3. $2m^2 + km + 32 = 0$
4. $kx^2 - 12x + 2k + 1 = 0$
5. $4x^2 + kx + k + 5 = 0$
6. $(2k + 5)y^2 - ky + 1 = 0$

7. For what value of m are the roots of $x^2 + 6x + m = 0$ imaginary?

8 For what values of k does the graph of $6x^2 + kx = 15$ intersect the x-axis?

9. One root of a quadratic equation is $1 + i$ Find the other root

The Product and Sum of the Roots [A]

Let x_1 and x_2 denote the two roots of

$$ax^2 + bx + c = 0 \qquad (1)$$

Then

$$x = x_1 \quad \text{and} \quad x = x_2$$
$$x - x_1 = 0 \quad \text{and} \quad x - x_2 = 0$$
$$(x - x_1)(x - x_2) = 0$$
$$x^2 - x(x_1 + x_2) + x_1 x_2 = 0 \qquad (2)$$

Dividing both members of equation (1) by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (3)$$

Equating the coefficients of equations (2) and (3), we have

$$x_1 + x_2 = \frac{-b}{a}$$

and

$$x_1 x_2 = \frac{c}{a}$$

These facts can be stated thus

The sum of the roots of a quadratic equation in x equals the negative of the coefficient of x divided by the coefficient of x^2

The product of the roots of a quadratic equation in x equals the constant term divided by the coefficient of x^2.

Can you prove these facts by the quadratic formula?

Example Find the sum and the product of the roots of $7x^2 + 19x - 6 = 0$

Solution In the equation $7x^2 + 19x - 6 = 0$, $a = 7$, $b = 19$, and $c = -6$

The sum of the roots $= -\frac{b}{a} = -\frac{19}{7}$.

The product of the roots $= \frac{c}{a} = \frac{-6}{7}$

[A] EXERCISES

Find the sum and product of the roots of each equation by the method used in the example above. Then check your answers by solving the equations and adding and multiplying the roots.

1. $x^2 - 5x + 6 = 0$	9. $3x^2 - 2x = 16$
2. $2y^2 - 3y + 2 = 0$	10. $4x^2 - 9x = 0$
3. $6x^2 - x - 3 = 0$	11. $5x^2 - 8x = 0$
4. $x^2 - 6x + 9 = 0$	12. $4x^2 - 100 = 0$
5. $3y^2 - 7y = 2$	13. $x^2 + 2bx - b^2 = 0$
6. $8x^2 + 14x - 15 = 0$	14. $2x^2 + 8cx + 6c^2 = 0$
7. $3y^2 + 11y = -6$	15. $3x^2 - 10mx + 3m^2 = 0$
8. $x^2 - x + 1 = 0$	16. $c^2x^2 + cx = 2$

Without solving the equations, tell which of the following solutions are correct.

17. $3x^2 + 5x - 2 = 0$. The roots are $\frac{1}{3}$ and -2.

18. $4x^2 - 6x + 5 = 0$. The roots are $\frac{1}{2}$ and $\frac{5}{2}$.

19. $2x^2 + 9x = 35$. The roots are $-3\frac{1}{2}$ and -1.

20. $5m^2 + 13m = 6$. The roots are $\frac{2}{5}$ and -3.

21. One root of $3x^2 + 7x = 40$ is -5. Find the other root.

22. One root of the quadratic equation $x^2 - 2x + 4 = 0$ is $1 + \sqrt{-3}$. Find the other root.

23. Find the value of n if one root of $6x^2 + 5x = n$ is $\frac{1}{2}$.

24. Solve the equation $2x^2 - 5x = 3$ graphically and tell what is needed to complete the following sentences which refer to the equation and its graph.

a. $\frac{5}{2}$ is the sum of the --?-- of the equation.

b. $\frac{5}{4}$, which is $\frac{-b}{2a}$, is --?-- the sum of the roots of the equation.

c. $\frac{5}{4}$ is the value of x at the --?-- --?-- of the graph.

d. For the roots of the equation to be equal, the graph must be raised --?-- units.

e. The axis of symmetry of the parabola intersects the x-axis in the point $x =$ __?__

f $4x = 5$ is the equation of the __?__ __?__ __?__.

Example Find the value of k if one root of the equation $2x^2 + x = k$ is -2

Solution The equation should be written $2x^2 + x - k = 0$ Let $r =$ one of the roots Then $r - 2 = -\frac{1}{2}$ Solving, $r = \frac{3}{2}$ The product of the roots is $-\frac{k}{2}$. Then $-\frac{k}{2} = (\frac{3}{2})(-2)$ and $k = 6$

25 Find the value of k if one root of $3y^2 + 2y + k = 0$ is 1.

Suggestion First find the other root

26 Find the value of c in the equation $3x^2 + cx - 8 = 0$ so that one root is 4

27. Find the value of k in the equation $y^2 + 9y + k = 0$ so that one root is twice the other

28 Find the value of m in the equation $10x^2 + mx = 3$ if one root is $\frac{1}{5}$

Forming Equations When the Roots Are Given [A]

The sum of the roots of the equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, and the product of the roots is $\frac{c}{a}$. These two facts enable one to form an equation when its two roots are known

Example Form the equation whose roots are 3 and $-\frac{2}{3}$

Solution 1. The sum of the roots $= 3 + \frac{-2}{3} = \frac{7}{3}$, and the product of the roots $= 3\left(\frac{-2}{3}\right) = -2$ Then $-\frac{b}{a} = \frac{7}{3}$ and $\frac{c}{a} = -2$ Solving the first equation for b and the second for c, we get $b = \frac{-7a}{3}$ and $c = -2a$ Substituting these values of b and c in $ax^2 + bx + c = 0$, we get $ax^2 - \frac{7a}{3}x - 2a = 0$ Clearing the last equation of fractions and dividing by a, we obtain the required equation, $3x^2 - 7x - 6 = 0$

Solution 2

$$
\begin{aligned}
x = 3 &\text{ and } x = -\tfrac{2}{3} \\
x - 3 = 0 &\text{ and } x + \tfrac{2}{3} = 0 \\
(x-3)(x+\tfrac{2}{3}) &= 0 \\
x^2 - \tfrac{7}{3}x - 2 &= 0 \\
3x^2 - 7x - 6 &= 0
\end{aligned}
$$

EXERCISES

[A]

Form the equations whose roots are

1 2 and 3
2 -4 and -6
3 6 and 6
4 5 and -5
5 2 and 8
6 $\frac{1}{2}$ and $\frac{1}{3}$
7 $\frac{1}{3}$ and $-\frac{1}{4}$
8 $\frac{1}{5}$ and $-\frac{1}{6}$
9 3 and 4
10 1.25 and 4
11 a and $-\frac{1}{2}a$
12 b and $\frac{2}{3}b$

Form the following equations

13 If the product of the roots is $\frac{2}{3}$ and the sum of the roots is $\frac{7}{3}$

14 If the sum of the roots is 1 and the product of the roots is 1

Form the equations whose roots are

15 $\sqrt{2}$ and $-\sqrt{2}$
16 i and $-i$
17 $i\sqrt{2}$ and $-i\sqrt{2}$
18 $\sqrt{5}-1$ and $\sqrt{5}+1$
19 $1-\sqrt{7}$ and $1+\sqrt{7}$
20 $4+i$ and $4-i$

[B]

Form the cubic equations whose roots are

21 2, 3, and -1
22 4, -6, and -2
23 -1, $+1$, and 3
24 -2, $1+i$, and $1-i$

25 $1, \dfrac{-1+3i}{2}$, and $\dfrac{-1-3i}{2}$

26 $-3, \dfrac{-3+3i\sqrt{3}}{2}$, and $\dfrac{-3-3i\sqrt{3}}{2}$

Equations in Quadratic Form [A]

Some equations that are not of the second degree can be written in quadratic form and solved by one of the methods used in solving quadratic equations. Four examples and their solutions will be given

Example 1 Solve $5x^4 - 125 = 0$

Solution
$$5x^4 - 125 = 0$$
$$5x^4 = 125$$
$$x^4 = 25$$
R₂ $$x^2 = \pm 5$$

$x^2 = +5$	$x^2 = -5$
R₂ $x = \pm\sqrt{5}$	R₂ $x = \pm i\sqrt{5}$

PROOF Does $5(+5)^2 - 125 = 0$? Yes
Does $5(-5)^2 - 125 = 0$? Yes
Does $5(i\sqrt{5})^2 - 125 = 0$? Yes
Does $5(-i\sqrt{5})^2 - 125 = 0$? Yes

Example 2 Solve $x^4 - 2x^2 - 24 = 0$

Solution This equation can be written $(x^2)^2 - 2(x^2) - 24 = 0$, which is a quadratic equation in x^2. Factoring, we have
$$(x^2 - 6)(x^2 + 4) = 0$$

$x^2 - 6 = 0$	$x^2 + 4 = 0$
$x^2 = 6$	$x^2 = -4$
$x = \pm\sqrt{6}$	$x = \pm 2i$

The roots are $\sqrt{6}$, $-\sqrt{6}$, $+2i$, and $-2i$. These roots may be proved by substituting in the original equation.

Example 3 Solve $(x^2 - x)^2 - (x^2 - x) - 30 = 0$

Solution This is a quadratic equation in $x^2 - x$. We shall let $y = x^2 - x$. Then the equation becomes
$$y^2 - y - 30 = 0$$
Factoring $$(y - 6)(y + 5) = 0$$
Replacing y by $x^2 - x$, we get
$$(x^2 - x - 6)(x^2 - x + 5) = 0$$
Setting each factor equal to zero, we get
$$x^2 - x - 6 = 0 \quad \text{and} \quad x^2 - x + 5 = 0$$
Solving these two equations, we have
$$x = 3 \quad x = -2, \quad x = \frac{1 + \sqrt{-19}}{2}, \quad \text{or} \quad x = \frac{1 - \sqrt{-19}}{2}$$

PROOF Does $(9 - 3)^2 - (9 - 3) - 30 = 0$? Yes
Does $(4 + 2)^2 - (4 + 2) - 30 = 0$? Yes
If $x = \dfrac{1 \pm \sqrt{-19}}{2}$, $x^2 - x = -5$
Does $(-5)^2 - (-5) - 30 = 0$? Yes

Example 4 Solve $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$

Solution This is a quadratic equation in $x^{\frac{1}{4}}$ Factoring the left member, we get $(x^{\frac{1}{4}} - 2)(x^{\frac{1}{4}} - 3) = 0$

$x^{\frac{1}{4}} - 2 = 0$	$x^{\frac{1}{4}} - 3 = 0$
$x^{\frac{1}{4}} = 2$	$x^{\frac{1}{4}} = 3$
$x = 16$	$x = 81$

The proof is left to the student

EXERCISES

[A]

Solve the following equations

1 $x^4 - 13x^2 + 36 = 0$

2 $y^4 - 256 = 0$

3 $x^4 - 5x^2 + 4 = 0$

4 $y^4 - 10y^2 + 9 = 0$

5 $3m^4 = 5m^2 + 2$

6 $y^4 - 6y^2 + 5 = 0$

7 $m^4 - 7m^2 + 6 = 0$

8 $y^4 = 13y^2 - 40$

9 $6x^4 = 7x^2 - 2$

10 $x^4 - 64 = 12x^2$

11 $(x+1)^2 - 3(x+1) = 40$

12 $(y-3)^2 + 4(y-3) - 21 = 0$

13 $(x^2 - x)^2 - 8(x^2 - x) = 9$

14 $(y^2 - y)^2 + 4(y^2 - y) = 12$

Solve and check

15 $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 10 = 0$

16 $y + \sqrt{y} - 6 = 0$

17 $x - \sqrt{x} = 30$

18 $x^{\frac{1}{3}} - 3x^{\frac{1}{6}} + 2 = 0$

19 $x^{-2} + x^{-1} - 3 = 0$

20 $2x^{-2} + 5x^{-1} + 3 = 0$

21 $\sqrt{x} - 3\sqrt[4]{x} + 2 = 0$

22 $3\sqrt{x} - 5\sqrt[4]{x} + 2 = 0$

Example 5 Find the three cube roots of 27

Solution Let x = the cube root of 27

Then
$$x^3 = 27$$
$$x^3 - 27 = 0$$
$$(x-3)(x^2 + 3x + 9) = 0$$

If $x - 3 = 0$	If $x^2 + 3x + 9 = 0$
$x = 3$	$x = \dfrac{-3 \pm 3i\sqrt{3}}{2}$

Then the roots are 3, $\dfrac{-3 + 3i\sqrt{3}}{2}$, and $\dfrac{-3 - 3i\sqrt{3}}{2}$

23 Find the three cube roots of 8

24 Find the three cube roots of -8

25. Find the three cube roots of 1 and check your solution

Checking Your Understanding of Chapter 15

Now you should check your understanding of this chapter. Be sure that you know

Review if you should

	PAGE
1 The formula for the solution of the quadratic equation $ax^2 + bx + c = 0$	407
2 That if $b^2 - 4ac = 0$, the roots are real, rational, and equal	407
3 That if $b^2 - 4ac$ is positive and a perfect square, the roots are real, rational, and unequal	408
4 That if $b^2 - 4ac$ is positive but not a perfect square, the roots are real, irrational, and unequal	408
5 That if $b^2 - 4ac$ is negative, both roots are imaginary	408
6 That the sum of the roots of $ax^2 + bx + c = 0$ is $\frac{-b}{a}$ and that the product of the roots is $\frac{c}{a}$	410
7 How to form an equation when its roots are known	412
8 How to solve equations in quadratic form	413

CHAPTER REVIEW

(A)

1 What is the discriminant of $ax^2 + bx + c = 0$?

2. What is the value of the discriminant of $ax^2 + bx + c = 0$ if the graph of the equation (a) intersects the x-axis in two points? (b) is tangent to the x-axis? (c) does not touch or intersect the x-axis?

3 What is the relation of a, b, and c when the roots of $ax^2 + bx + c = 0$ are (a) real and equal? (b) rational and unequal? (c) imaginary? (d) irrational and unequal?

4 What are the two roots of the equation $ax^2 + bx + c = 0$?

5. What is the name of the graph of $y = ax^2 + bx + c$?

Give the sum and product of the roots of each of the following equations

6 $x^2 - 2x + 8 = 0$

7. $12x^2 + 11x = 5$

8 $x^2 + x - 1 = 0$

9 $2x^2 + 7x = 4$

10 $2y^2 - 30 = -3y^2$

11 $13y - 30 = -3y^2$

Give the character of the roots of

12 $y^2 - 4y - 21 = 0$

13 $4x^2 - 28x + 49 = 0$

14 $7x^2 - 175 = 0$

15 $2x^2 - x + 5 = 0$

Form the equations whose roots are

16 2 and -1

17 $\sqrt{2}$ and $-\sqrt{2}$

18 One root of a quadratic equation is $1 - \sqrt{-1}$ Find the other root

19 Solve $x^4 - 11x^2 + 28 = 0$

20 Solve $2(x^2 + x)^2 - 7(x^2 + x) = 30$

21 Solve $2x^{\frac{2}{3}} - x^{\frac{1}{3}} = 15$

CHAPTER TEST

[A]

Find the values of the discriminants of

1 $x^2 - 6x + 9 = 0$

2 $3x^2 - x + 7 = 0$

Describe the roots of

3 $2x^2 + 5x - 12 = 0$

4 $9x^2 + 1 = -6x$

5 What do you know of the roots of $ax^2 + bx + c = 0$ if $b^2 - 4ac$ is negative?

Form the equations whose roots are

6 -2 and 5

7 $\frac{1}{2}$ and $\frac{2}{3}$

8 $\sqrt{3}$ and $2\sqrt{3}$

9 $4i$ and $-4i$

Find the sum of the roots of

10 $3x^2 - 4x + 2 = 0$

11 $y^2 - 7y + 8 = 0$

12 $2h^2 - 10h = 32$

13 $5x^2 + 15x - 70 = 0$

Find the product of the roots of

14 $4x^2 - 18x - 24 = 0$

15 $2m^2 - 9m = -8$

16 For what value of k does the equation $4x^2 - 12x + k = 0$ have equal roots?

17 One root of the equation $6x^2 + 2x - p = 0$ is $-\frac{1}{3}$ Find the other root

18 One root of the equation $4x^2 + px + 12 = 0$ is 3 What is the other root?

19 Solve $x^4 - 17x^2 = -16$

20 Solve $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 9$

CHAPTER 16

Determinants

In this chapter you will learn to use a new symbol.

Meaning of a Determinant [A]

A determinant is a square array like

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

having as many rows as it has columns The number of rows or columns determines the order of the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is a determinant of the *second order* and represents the expression $a_1b_2 - a_2b_1$ The numbers a_1, b_1, a_2, and b_2 are called the elements of the determinant The elements a_1 and b_2 form the principal diagonal The value of a determinant of the second order is found by subtracting from the product of the two elements in the principal diagonal the product of the two elements in the other diagonal

Example Evaluate $\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix}$

Solution $\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} = (2)(5) - (3)(-4) = 22$

The Solution of Two Linear Equations in Standard Form [A]

Let us solve a set of two linear equations in standard form

$$a_1x + b_1y = c_1 \qquad (1)$$
$$a_2x + b_2y = c_2 \qquad (2)$$

M (1) by b_2 $\qquad a_1b_2x + b_1b_2y = c_1b_2$

M (2) by b_1 $\qquad \underline{a_2b_1x + b_1b_2y = c_2b_1}$

Subtracting $\qquad a_1b_2x - a_2b_1x = c_1b_2 - c_2b_1$

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1$ is not zero

In a similar way x may be eliminated from the two equations to obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1$ is not zero

Study these values of x and y carefully It was the symmetrical arrangement of the terms in the values that led to the introduction of determinants

We notice that the denominators of the values of x and y are the same We shall represent the denominator by

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

This determinant is called the determinant of the coefficients

The numerator for the value of x may be represented by

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

The determinant for D_x may be obtained from D by replacing the coefficients of x by the corresponding c's In a similar way the numerator for the value of y may be written

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

To Solve a Set of Two Linear Equations in Two Variables in Standard Form

1 Evaluate the determinants D, D_x, and D_y.

2 Then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $D \neq 0$

Example Solve by determinants $2x - y = 10$
$5x + 2y = 7$

Solution
$$D = \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = 4 + 5 = 9$$
$$D_x = \begin{vmatrix} 10 & -1 \\ 7 & 2 \end{vmatrix} = 20 + 7 = 27$$
$$D_y = \begin{vmatrix} 2 & 10 \\ 5 & 7 \end{vmatrix} = 14 - 50 = -36$$

Then $x = \frac{D_x}{D} = \frac{27}{9} = 3$ and $y = \frac{D_y}{D} = \frac{-36}{9} = -4$

PROOF $2x - y = 10$ | $5x + 2y = 7$
Does $6 + 4 = 10$? Yes | Does $15 - 8 = 7$? Yes

If the determinant

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = 0$$

we have from $a_1b_2 - a_2b_1 = 0$, $a_1b_2 = a_2b_1$, or, writing it as a proportion,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

This means that the equations are either equivalent or inconsistent and there is no solution For example,

$$\left.\begin{matrix} 3x+3y=1 \\ 5x+5y=2 \end{matrix}\right\} \text{Inconsistent} \qquad \left.\begin{matrix} 3x+3y=6 \\ 5x+5y=10 \end{matrix}\right\} \text{Equivalent}$$

[A] **EXERCISES**

Evaluate

1 $\begin{vmatrix} 6 & -2 \\ 3 & -5 \end{vmatrix}$ 2 $\begin{vmatrix} -1 & 0 \\ -4 & 5 \end{vmatrix}$ 3 $\begin{vmatrix} 9 & -8 \\ 3 & 0 \end{vmatrix}$ 4 $\begin{vmatrix} e^{x} & \frac{1}{2} \\ e^{-x} & \frac{1}{2} \end{vmatrix}$

Solve by determinants

5 $4x-y=5$
$3x+2y=12$

6. $3x+2y=1$
$2x-3y=18$

7. $5x-2y=19$
$7x+3y=15$

8 $2x+3y=2$
$5x-y=-29$

9 $4x-3y=8$
$4x-5y=0$

10 $3x+7y=17$
$y=2x$

[B] **EXERCISES**

1 $\frac{5y}{4}+\frac{7x}{8}=51$
$\frac{y}{4}+\frac{7x}{4}=12$

3 $\frac{1}{x}+\frac{3}{y}=5$
$\frac{3}{x}-\frac{4}{y}=2$

2 $\frac{7(x-5)}{3}+\frac{x}{3}=1-\frac{y}{6}$
$\frac{x-y}{4}+\frac{y}{2}=\frac{5(x-1)}{3}-1$

4 $\frac{m}{r}+\frac{n}{y}=h$
$\frac{r}{x}+\frac{s}{y}=k$

5 Prove that $\begin{vmatrix} ta_1 & b_1 \\ ta_2 & b_2 \end{vmatrix}=t\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

6 Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=-1\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$

7. Prove that $\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix}=0$

Find the value of the following

8. $\begin{vmatrix} 2a & -3a \\ -a & a \end{vmatrix}$

9. $\begin{vmatrix} \sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{2} \end{vmatrix}$

10 $\begin{vmatrix} (a-b) & (a+b) \\ (a+b) & (a-b) \end{vmatrix}$

11 $\begin{vmatrix} t & -t \\ t^2 & t \end{vmatrix}$

12. $\begin{vmatrix} \sqrt{x-1} & 1 \\ 1 & \sqrt{x-1} \end{vmatrix}$

13 $\begin{vmatrix} 2\sqrt{3} & t \\ 4t & \sqrt{27} \end{vmatrix}$

Determinants of the Third Order [A]

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

having 3 rows and 3 columns, is a determinant of the *third order* It represents the expression $a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$, which is called the expansion of the determinant $a_1b_2c_3$ is the principal diagonal

The expansion just given can be written by different methods One method is as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

1 Repeat the first two columns of the determinant at the right of the determinant

2 Draw arrows through the diagonals as shown

3 Find the product of each set of three elements through which an arrow is drawn, giving each product a plus sign if the arrow points down and a minus sign if the arrow points up

This method of evaluating a third-order determinant does not apply to any other higher order

Example Evaluate $\begin{vmatrix} 2 & -3 & 4 \\ 5 & -1 & -2 \\ 6 & 0 & -7 \end{vmatrix}$

Solution $\begin{vmatrix} 2 & -3 & 4 \\ 5 & -1 & -2 \\ 6 & 0 & -7 \end{vmatrix} \begin{matrix} 2 & -3 \\ 5 & -1 \\ 6 & 0 \end{matrix}$

$$= (2)(-1)(-7) + (-3)(-2)(6) + (4)(5)(0) - (6)(-1)(4) - (0)(-2)(2) - (-7)(5)(-3)$$
$$= 14 + 36 + 0 + 24 - 0 - 105 = -31$$

EXERCISES [A]

Evaluate

1 $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 5 & 6 \end{vmatrix}$

2 $\begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix}$

3 $\begin{vmatrix} a & 0 & 1 \\ b & 1 & 0 \\ c & c & 1 \end{vmatrix}$

4 $\begin{vmatrix} 3 & -1 & 5 \\ 2 & -2 & 3 \\ 1 & -3 & 7 \end{vmatrix}$ 5 $\begin{vmatrix} -1 & 1 & -1 \\ -2 & 2 & -6 \\ 3 & -3 & 4 \end{vmatrix}$ 6 $\begin{vmatrix} 3 & 0 & -7 \\ -5 & 0 & 2 \\ 6 & 3 & 1 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 5 & 3 & -1 \end{vmatrix}$ 8 $\begin{vmatrix} x & y & 1 \\ 2 & -3 & 1 \\ -4 & 5 & 1 \end{vmatrix}$ 9 $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution of Sets of Linear Equations in Three Variables [A]

If we were to solve the following set of equations in standard form,

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$
$$a_2x + b_2y + c_2z = d_2 \tag{2}$$
$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

by the method of addition and subtraction, we should find that

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

and $$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

We notice that the denominators for x, y, and z are all equal, and may be expressed by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A study of the numerator for x reveals that the numerator is the same as the denominator except that the a's are replaced by d's Therefore we may write

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

In like manner,

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \text{and} \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Hence the solution of the given set of equations is

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ D \neq 0$$

To Solve a Set of Three Linear Equations in Three Variables

1. Find D, the determinant of the coefficients
2. To find D_x, replace the coefficients of x in D by the corresponding constant terms
3. To find D_y, replace the coefficients of y in D by the corresponding constant terms.
4. To find D_z, replace the coefficients of z in D by the corresponding constant terms
5. Then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{D}$, $D \neq 0$

Example Solve
$$\begin{aligned} 2x - 3y - 4z &= 4 \\ 3x - 4y &= 1 \\ 5x - 3y + 9z &= 0 \end{aligned}$$

Solution

$$D = \left|\begin{array}{ccc|cc} 2 & -3 & -4 & 2 & -3 \\ 3 & -4 & 0 & 3 & -4 \\ 5 & -3 & 9 & 5 & -3 \end{array}\right| = -72 + 0 + 36 - 80 - 0 + 81 = -35$$

$$D_x = \left|\begin{array}{ccc|cc} 4 & -3 & -4 & 4 & -3 \\ 1 & -4 & 0 & 1 & -4 \\ 0 & -3 & 9 & 0 & -3 \end{array}\right| = -144 + 0 + 12 - 0 - 0 + 27 = -105$$

$$D_y = \left|\begin{array}{ccc|cc} 2 & 4 & -4 & 2 & 4 \\ 3 & 1 & 0 & 3 & 1 \\ 5 & 0 & 9 & 5 & 0 \end{array}\right| = 18 + 0 + 0 + 20 + 0 - 108 = -70$$

$$D_z = \left|\begin{array}{ccc|cc} 2 & -3 & 4 & 2 & -3 \\ 3 & -4 & 1 & 3 & -4 \\ 5 & -3 & 0 & 5 & -3 \end{array}\right| = 0 - 15 - 36 + 80 + 6 + 0 = 35$$

Then
$$x = \frac{D_x}{D} = \frac{-105}{-35} = 3$$
$$y = \frac{D_y}{D} = \frac{-70}{-35} = 2$$
$$z = \frac{D_z}{D} = \frac{35}{-35} = -1$$

The proof is left to the student

In the solution of a set of linear equations in three variables if $D = 0$ and the numerator determinants are not equal to 0 there are at least two inconsistent equations if $D = 0$ and all the numerator determinants are 0 there are at least two equivalent equations

[A] **EXERCISES**

Solve by determinants

1. $3x - 2y - 4z = 15$
   $4x + 2y - 5z = 18$
   $x - 3y - 6z = 5$

2. $3x - 2y = 7$
   $3y - 4z = 6$
   $5x - 2z = 11$

3. $3x - 2y + 5z = 15$
   $4x + y - z = 9$
   $x - 5y + 2z = 18$

4. $x + y + z = 6$
   $x + y - z = 0$
   $x - y - z = 2$

5. $x + y = 4$
   $x + z = -2$
   $y + z = 8$

6. $x + 5y + 3z = 4$
   $3x - 2y + 4z = 21$
   $2x + 3y - z = -13$

7. $5x - y + 3z = 16$
   $7x + 6y + z = 34$
   $2x + 5y - 7z = 9$

8. $x + y + 3z = 11$
   $x + 3y + z = 11$
   $3x + y + z = 13$

Minors [A]

The minor of any given element of a determinant is the determinant that remains when the column and the row containing the given element are deleted For example, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

the minor of a_2 is $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$

the minor of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

and so on

Expansion of a Determinant by Minors [A]

The methods of expansion given on pages 419 and 422 for second-order and third-order determinants, respectively, are special methods that apply to those orders only The following method applies to a determinant of any order

Expansion by Minors

The value of a determinant
is the algebraic sum of the products
of each element in any row (or column) by its corresponding minor,
the sign of each product being positive or negative
according to whether the sum of the number of the row and
number of the column containing the element is even or odd

The rows of a determinant are numbered from top to bottom and the columns from left to right In the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$a_1b_1c_1$ is the first row, $a_1a_2a_3$ is the first column and so on

Example Expand by minors

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solution 1 We may select any row or column as the one in terms of whose minors the determinant is to be expanded Suppose we choose the first column

2 Since a_1 is in the first row and first column ($1 + 1 = 2$, an even number), the sign of the product is $+$ Then the product of a_1 and its minor is

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

3 Since a_2 is in the second row and first column ($2 + 1 = 3$, an odd number), the sign of the product is $-$ Then the product of a_2 and its minor is

$$-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

4 Since a_3 is in the third row and first column ($3 + 1 = 4$, an even number) the sign of the product is $+$ Then the product of a_3 and its minor is

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

5 Therefore the value of the determinant is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

This value is identical with the value found by the expansion of the same determinant on page 422

Notice in this expansion that a third-order determinant is reduced to the sum of three second-order determinants If we expand a fourth-order determinant by minors, we first obtain the sum of four third-order determinants Then, expanding each of the third-order determinants by minors, we obtain the sum of twelve second-order determinants This seems like a long and tedious process However, there are some important properties of determinants that materially lessen the computation involved

Important Properties of Determinants [A]

In the following theorems we shall use determinants of the third order However, the theorems are valid for determinants of any order

Theorem 1 **The sign of a determinant is changed by interchanging two rows or two columns**

Thus
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

This can be verified by expanding the two determinants

Theorem 2 **The value of a determinant is unchanged if corresponding rows and columns are interchanged**

Thus
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This can be verified by expanding both determinants

Theorem 3 **The value of a determinant is zero if each of the elements of a row (or column) is zero**

Thus
$$\begin{vmatrix} a & 0 & r \\ b & 0 & s \\ c & 0 & t \end{vmatrix} = 0$$

If the preceding determinant is expanded by minors in terms of the second column, each term of the expansion will be zero, and therefore the value of the determinant will be zero

Theorem 4 The value of a determinant is zero if any two rows or columns are identical

Thus $$\begin{vmatrix} a & r & a \\ b & s & b \\ c & t & c \end{vmatrix} = 0$$

Since the first and third columns are identical, interchanging them does not affect the value of the determinant But Theorem 1 asserts that the value changes sign Therefore the value of the determinant equals the negative of itself and must be zero

Theorem 5 If each element in any row (or column) of a determinant is multiplied by m, the value of the determinant is multiplied by m

Thus $$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This can easily be verified by expanding the two determinants

Theorem 6 The value of a determinant is unchanged if the elements of any row (or column) are multiplied by any quantity and added to (or subtracted from) the corresponding elements of another row (or column)

Thus $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} (a_1 - mc_1) & b_1 & c_1 \\ (a_2 - mc_2) & b_2 & c_2 \\ (a_3 - mc_3) & b_3 & c_3 \end{vmatrix}$$

In this illustration the elements of the third column were multiplied by m and subtracted from the corresponding elements in the first column This theorem can be verified by expanding by minors, using elements of the first column

Example 1 Evaluate $\begin{vmatrix} 4 & 2 & 4 \\ 4 & 2 & 6 \\ 3 & 1 & 5 \end{vmatrix}$

Solution If we multiply each element of the second column by 2 and subtract the product from the corresponding element of the first column, we obtain

$$\begin{vmatrix} 0 & 2 & 4 \\ 0 & 2 & 6 \\ 1 & 1 & 5 \end{vmatrix}$$

If we expand this by minors using the elements of th
we have

$$\begin{vmatrix} 0 & 2 & 4 \\ 0 & 2 & 6 \\ 1 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 2 & 6 \end{vmatrix} = 4$$

Example 2 Evaluate $\begin{vmatrix} 6 & 7 & -2 \\ -9 & 4 & 3 \\ 3 & 12 & -1 \end{vmatrix}$

Solution If we multiply each element of the third column by 3 and add to the corresponding element in the first column we obtain a determinant in which all the elements of the first column are zero Then by Theorem 3 the value of the determinant is zero

EXERCISES

[A]

Evaluate the following determinants

1 $\begin{vmatrix} 6 & 0 & 7 \\ -3 & 2 & 1 \\ 4 & 9 & 3 \end{vmatrix}$ 2 $\begin{vmatrix} -1 & 3 & 4 \\ 2 & -3 & -8 \\ 6 & 5 & 10 \end{vmatrix}$ 3 $\begin{vmatrix} 5 & 0 & 2 \\ -5 & 3 & 7 \\ 10 & 6 & 4 \end{vmatrix}$

4 Solve by determinants $3x + 2y - z = 14$
$x - y + z = -3$
$2x - y = 1$

[B]

5 $\begin{vmatrix} -2 & 3 & 0 & 4 \\ 7 & 3 & 4 & 0 \\ 5 & -6 & 2 & 1 \\ 4 & -6 & 0 & -8 \end{vmatrix}$ 8 $\begin{vmatrix} 1 & 4 & 4 & 4 \\ 4 & 1 & 4 & 4 \\ 4 & 4 & 1 & 4 \\ 4 & 4 & 4 & 1 \end{vmatrix}$

6 $\begin{vmatrix} -1 & 3 & -2 & 1 \\ 0 & 2 & 8 & 6 \\ 7 & 4 & 5 & 3 \\ 2 & 3 & 0 & 1 \end{vmatrix}$ 9 $\begin{vmatrix} 4 & -3 & 2 & 1 \\ 7 & 5 & -3 & 0 \\ 8 & -3 & 4 & 2 \\ -24 & 18 & -12 & -6 \end{vmatrix}$

7 $\begin{vmatrix} 3 & -5 & 2 & 6 \\ 7 & 0 & 0 & 3 \\ 6 & 2 & -5 & 3 \\ -7 & 2 & 8 & -6 \end{vmatrix}$ 10 $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 2 & 3 & 0 & 5 \\ 3 & 4 & 5 & 0 \end{vmatrix}$

Solve the following systems of equations by determinants

11 $x - 3y + 2z - 2u = 0$
$3x - 2y + z = -7$
$5x - 2y - 5u = 9$
$4x + 5z + 8u = 1$

12 $3x - 2y = 12$
$2y - 7z = 6$
$4z - 3u = 6$
$x - 5u = 16$

If the precedingy **Determinants** [8]

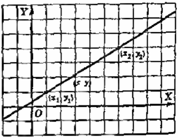
second column the value of th gnize $\dfrac{y-y_1}{x-x_1}=\dfrac{y_2-y_1}{x_2-x_1}$ as the equation of the straight

Theorem 4 Th oints (x_1, y_1) and (x_2, y_2)? Clear the equation of frac

are identico nplify it

ite the expansion of

Thus
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$



Observe that the equation you obtain when you clear $\dfrac{y-y_1}{x-x_1}=\dfrac{y_2-y_1}{x_2-x_1}$ of fractions is the same as the expansion of the determinant, in other words, observe that the determinant provides a way to write the equation of a line through two points. Thus

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 5 & 4 & 1 \end{vmatrix}=0$$

is the equation of the line through points (1, 2) and (5, 4)

[8]

EXERCISES

With determinants write the equation of the line through the following points. Then expand each determinant to get the equation in standard form

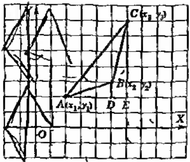
1. $(-3, -4)$ and $(3, 2)$
2. $(6, 1)$ and $(-3, 5)$
3. $(7, 2)$ and $(-2, 4)$

Solve each of the following equations

4 $\begin{vmatrix} x & 1 \\ 8 & 2 \end{vmatrix}=0$ 5 $\begin{vmatrix} x & x & 1 \\ 3 & 1 & 5 \\ 1 & 2 & 6 \end{vmatrix}=0$ 6 $\begin{vmatrix} x & 0 & 1 \\ 3 & 2 & 1 \\ 4 & 0 & 5 \end{vmatrix}=0$

Area of Triangle Whose Vertices Are Known [8]

Determinants provide a simple way to find the area of a triangle when the co-ordinates of its vertices are known. To discover the procedure, answer the following questions



1 Do you see that
the area of $\triangle ABC = \triangle AEC - \triangle ADB -$ trapezoid $DECB$?

2 Do you see that
the area of $\triangle AEC = \frac{1}{2} AE \cdot EC = \frac{1}{2}(x_3 - x_1)(y_3 - y_1)$?

3 Do you see that
the area of $\triangle ADB = \frac{1}{2} AD \cdot DB = \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$?

4 Do you see that
the area of trapezoid $DECB = \frac{1}{2} DE(DB + EC)$
$= \frac{1}{2}(x_3 - x_2)[(y_2 - y_1) + (y_3 - y_1)] = \frac{1}{2}(x_3 - x_2)(y_2 - 2y_1 + y_3)$?

5 Do you see that if we substitute in the formula of (1) the values found in b, c and d, we obtain
$\triangle ABC$
$= \frac{1}{2}[(x_3 - x_1)(y_3 - y_1) - (x_2 - x_1)(y_2 - y_1) - (x_3 - x_2)(y_2 - 2y_1 + y_3)]$
$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3)$?

Do you see that this can be written in determinant form as

$$\text{area of } \triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Notice that the vertices of the triangle appear in the determinant in counterclockwise order

EXERCISES [8]

Find the area of each of the following triangles having given their vertices

1 (1, 1), (5, 2), (3, 6)
2 (−2, −1), (3, 2), (2, 3)
3 (2, 1), (−2, 3), (−3, −1)
4 (−2, −2), (−1, 3), (−5, −1)

Homogeneous Linear Equations [8]

A homogeneous equation is an equation in which all terms are of the same degree in the variables Thus $x^2y - 4xy^2 + x^2y^2 = 0$ is a

homogeneous equation of the fourth degree By definition a homogeneous equation cannot contain a constant term other than zero

A homogeneous linear equation is an equation in which all the terms are of the first degree $x + y - 3z = 0$ is a homogeneous linear equation

Example Let us consider the homogeneous linear system

$$\begin{aligned} x - y - z &= 0 \\ 3x - 2y - 5z &= 0 \\ 2x - y - 4z &= 0 \end{aligned}$$

Solution. It is obvious that $x = 0, y = 0, z = 0$, is always a solution of such a system This solution is known as the *zero* or *trivial* solution

It is also obvious that, if we solve any homogeneous linear system by determinants, the numerator determinants will all have a column of zeros, and therefore by Theorem 3 the numerators will all be zero If the determinant of the coefficients is not zero, we shall obtain only the trivial solution However, if the determinant of the coefficients is zero, we shall have each variable equal to $\frac{0}{0}$, which is indeterminate In this case there is an unlimited number of solutions besides the trivial solution

The determinant of the coefficients of the above system is

$$\begin{vmatrix} 1 & -1 & -1 \\ 3 & -2 & -5 \\ 2 & -1 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

Therefore there are other solutions of the system

Dividing the first two equations by z, we obtain

$$\frac{x}{z} - \frac{y}{z} = 1$$

$$3\frac{x}{z} - 2\frac{y}{z} = 5$$

Solving this system by determinants or otherwise, we obtain $\frac{x}{z} = 3$ and $\frac{y}{z} = 2$, or $x = 3z$ and $y = 2z$

If we let $z = 1$, we have $x = 3$, $y = 2$, and $z = 1$ as a nontrivial solution Other solutions may be obtained by assigning any value to z Thus if $z = -3$, we have $x = -9$, $y = -6$, and $z = -3$ as a solution If $z = \sqrt{2}$, we have $x = 3\sqrt{2}$, $y = 2\sqrt{2}$, and $z = \sqrt{2}$ as a solution

[B] **EXERCISES**

Find the nontrivial solutions, as in the example on page 432, for the following equations when such solutions exist

1. $x+y+z=0$
   $2x+5y-4z=0$
2. $x+y-2z=0$
   $x-2y-11z=0$
3. $x+y-z=0$
   $x-2y-4z=0$
   $2x+3y-z=0$
4. $3x-y-2z=0$
   $3x+4y+7z=0$
   $x-2y-5z=0$
5. $x-2y-z=0$
   $x+4y-4z=0$
   $2x+5y+z=0$
6. $x-y+z=0$
   $3x+y+z=0$
   $2x+y-2z=0$
7. $x+3y+10z=0$
   $3x+2y+9z=0$
   $4x-2y-2z=0$
8. $x+y-z-4w=0$
   $x-3y+5z-4w=0$
   $2x+3y+5z-9w=0$
   $3x+5y+z+2w=0$
9. $x+y-z-w=0$
   $x+y+z+w=0$
   $3x-2y+5z-3w=0$
   $7x+y-z+5w=0$
10. $2x+3y+2z+3w=0$
    $x-3y-2z+6w=0$
    $3x+2y-2z+2w=0$
    $3x+5y+4z-5w=0$

Checking Your Understanding of Chapter 16

Before you leave this chapter make sure that you

1 Know what a determinant is (p 419)

2 Can expand a determinant of the second order (p 419) and the third order (p 421) If you expect to continue the study of mathematics, you will also want to be sure you can expand determinants of higher order by the use of minors (p 425)

Review if you need to

3 Can use determinants to solve systems of equations (pp 419–424) If you expect to continue the study of mathematics, you will want to be able to find the nontrivial solutions, when they exist, for sets of homogeneous linear equations (pp 431–433)

4 Know the important properties of determinants (p 427)

5 Can spell and use correctly the following words and phrases

determinant (p 419)
expansion (p 422)
minor (p 425)
principal diagonal (p 419)

MATHEMATICS NEEDED IN THE STUDY OF MEDICINE AND IN MEDICAL RESEARCH

Some medical schools specify at least two years of college training as an entrance requirement, others require three years, and others require a college degree

Students looking forward to the study of medicine will find the following high school subjects desirable

English (4 years)	Physiology
Algebra ($1\frac{1}{2}$ years)	Physics
Plane geometry (1 year)	Latin (2 years)
Zoology	Social studies
Chemistry	

Mathematics has wide application in medicine Its use ranges from comparatively simple procedures, such as prescription writing, to complex and detailed studies such as the investigations of body mechanics made by orthopedic surgeons.

The question 'How much college mathematics is needed in the study of medicine? is difficult to answer This is true because medical schools have different courses of study and place different degrees of emphasis on like subjects For example, the amount of mathematics needed in physiological chemistry depends upon the intensity of the mathematics in any particular course One of the foremost schools of medicine has in its requirements for admission the following sentence In view of the increasing employment of mathematical concepts in the medical sciences students expecting to take up the study of medicine are advised to include one year of mathematics, calculus if possible, in their college course A student planning to enter the field of medical research should acquire as much mathematical education as possible

A young physician recently said that he found trigonometry useful when he studied physics in his premedical work

A prominent ear, eye, nose, and throat specialist said that he had had difficulty in some of his medical training because of his lack of understanding of logarithms.

A recent graduate of an outstanding medical school stated that one cannot know too much mathematics for the study of medicine He had studied mathematics in college for two or more years before entering the medical school

Abbott Laboratories

Both the research doctor and the physician realize the importance of mathematics

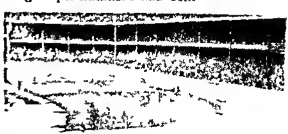
Harold Lambert

CHAPTER 17

Permutations, Combinations, and Probability

In this chapter we shall study the laws of chance and the ways of grouping objects

To obtain some idea of the meaning of permutations and combinations, suppose that a high-school baseball coach has three boys, Al, Bill, and Chuck, each of whom can pitch or catch. Let us see how many batteries (a battery consists of a pitcher and catcher) can be had with these three players. Without reference to who pitches and who catches, three different batteries can be formed. They are



FIRST BATTERY	SECOND BATTERY	THIRD BATTERY
Al and Bill	Al and Chuck	Bill and Chuck

These three groups (batteries) are combinations of 3 boys taken 2 at a time ($_3C_2$). Each of these three batteries can be arranged in two ways as follows:

FIRST BATTERY

Al	pitcher	Bill	pitcher
Bill	catcher	Al	catcher

SECOND BATTERY

Al	pitcher	Chuck	pitcher
Chuck	catcher	Al	catcher

THIRD BATTERY

Bill	pitcher	Chuck	pitcher
Chuck	catcher	Bill	catcher

These six groups are permutations of 3 boys taken 2 at a time ($_3P_2$). Each of the three combinations above has two permutations. There is no order in combinations, but there is order in permutations.

A combination is a group of objects in which the order or arrangement is not considered.

A permutation is an arrangement of a group of objects in a definite order.

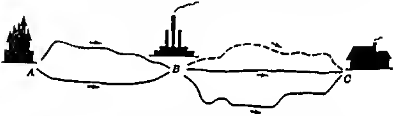
The discussion and problems of this chapter depend upon the following fundamental principle:

> If one act can be performed in any one of m different ways,
> and if after it has been done
> a second act can be done in n different ways,
> then the number of ways of performing the two acts in succession is mn.

This principle can be extended to include any number of acts

Example 1 If there are two roads from A to B and three roads from B to C, how many routes are there from A to C by the way of B?

Solution. The journey from A to B can be made in either of two ways After this part of the journey has been made, the remainder of the trip can be made in any one of three ways So for each of the 2 ways of going from A to B, there are 3 ways of completing the trip In all there are 2×3 ways, or 6 ways, of going from A to C



Example 2 A girl has 3 blouses and 4 skirts How many different costumes can she arrange with them?

Solution With each of the 3 blouses she can wear any one of the 4 skirts Then she can have 3×4, or 12, costumes

EXERCISES

[A]

1. There are 4 railroads between A and B, and 5 railroads joining B and C How many different routes can one take in going from A to C by way of B?

2. A boy has 3 pairs of trousers, 2 hats, and 4 coats In how many different costumes can he appear?

3. In how many different ways can Carl, John, and Bill stand in a straight line?

4. In how many different ways can volumes I, II, and III be arranged on a shelf?

5. In how many different ways can four books be arranged on a shelf?

6 There are four high schools in a city In how many different ways can three pupils attend these schools if no two of them attend the same school?

7 A building has 4 entrances and 3 exits How many different routes can a man take in going into the building and coming out of it?

8 How many different dancing couples, consisting of a boy and a girl, can be formed from a group of 7 boys and 8 girls?

9 A boy buys bread at one store and a ball at another He can buy bread at 8 different stores and a ball at 3 different stores If he buys a loaf of bread at one store and a ball at another, how many sets of two stores does he have available?

Permutations of Things All Different[A]

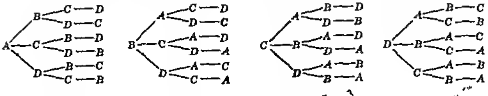
As you have learned, a permutation is an arrangement of objects in a definite order The number of permutations of 3 things taken 2 at a time is represented by $_3P_2$, the number of permutations of 6 things taken 4 at a time is represented by $_6P_4$, and the number of permutations of n things taken r at a time is represented by $_nP_r$

Let us determine how many permutations (arrangements) can be made with the 4 letters A, B, C, and D We can fill the first place in any one of 4 ways Let us choose A for the first place After we have selected A for the first place we have 3 letters left, and any one of them can be chosen for the second place Let us use B for the second place For the third place we can select either of the two remaining letters, C and D Let us choose C for the third place After the first three places are filled, there is only one way of filling the fourth place So D is used for the fourth place

Since there are 4 ways of filling the first place, then three ways of filling the second place, then two ways of filling the third place, then one way of filling the fourth place, the total number of 4 letters taken 4 at a time is $4 \times 3 \times 2 \times 1$, or 24

The solution is $_4P_4 = 4 \times 3 \times 2 \times 1 = 24$

The 24 permutations can be shown diagrammatically as follows



The first permutation is $ABCD$, the second is $ABDC$, the fifteenth is $CBAD$, and the twenty-third is $DCAB$

We shall now use this kind of reasoning to develop a formula for finding the number of permutations of n things taken r things at a time This means that we are to find all possible arrangements of n things when r of them are used at a time It means that all possible groups of r things each are to be selected from n things, and that each of these groups is to be arranged in all possible ways We shall speak of filling the r spaces with n things

The first place can be filled in n different ways Then there are $n - 1$ things left, and the second place can be filled in $n - 1$ different ways After it is filled, there are $n - 2$ things left, and the third place can be filled in $n - 2$ ways, etc After $r - 1$ places (all but the last one) have been filled, there are $n - (r - 1)$, or $n - r + 1$, things left So the last place can be filled in $n - r + 1$ ways Since each of the n ways can be followed by $n - 1$ ways, each of the $n - 1$ ways can be followed by $n - 2$ ways, and so on, we have

$$_nP_r = n(n-1)(n-2) \quad (n-r+1)$$

You should remember that the first factor in the formula above is n, and that there are r factors In using this formula you should remember that no two of the elements (things) are the same, and that no element can be used to fill more than one space

If all the elements are to be used in each permutation, $r = n$ Then the formula becomes $_nP_n = n(n-1) \quad 2 \quad 1$

$$_nP_n = n!$$

Example 1 How many three-digit positive integers can be made by using the digits 1, 2, 3, 4, and 5 if no digit is repeated in any number?

Solution Since no digit can be repeated in any number, a number such as 252 is excluded The hundreds place can be filled in 5 ways, then the tens' place can be filled in 4 ways and then the units place can be filled in 3 ways The number of positive integers $= 5 \times 4 \times 3 = 60$ If we use the formula, $_5P_3 = 5 \times 4 \times 3 = 60$

Example 2 In how many different ways can the letters of the word *PLANE* be arranged?

Solution. We are to find the number of permutations of 5 things taken 5 at a time

$$_5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Example 3 A football squad has 14 players If any of the 14 players can fill any one of the 11 positions on the team, how many different team arrangements can be placed on the field?

Solution $\qquad n = 14 \quad \text{and} \quad r = 11$

$$_{14}P_{11} = 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$$
$$= 14{,}529{,}715{,}200$$

Example 4 How many different batting orders can a baseball team have if only the last three places are changed?

Solution Only the last three places affect the number of permutations

$$_3P_3 = 3! = 6$$

EXERCISES

(A)

1 Find the value of $_6P_4$

2 Find the value of $_8P_3$

3 Find the value of $_9P_5$

4 Find the value of $_7P_7$

5 In how may ways can 8 books be arranged on a shelf?

6 How many different positive integers of 2 digits each can be made with the digits 2, 4, 5, and 8 if no digit is repeated in a number?

7 How many positive integers of 3 digits each can be formed with the digits 3, 4, 5, 6, and 7 if no digit is repeated in a number?

8 How many positive even integers can be formed with the digits 2, 5, 6, 7, and 8 if no digit is repeated in a number? (Do not use the formula Start the solution by filling the units' place.)

9 How many odd positive integers can be formed with the digits 3, 4, 5, 6, 7, and 8 if no digit is repeated in a number?

10 In how many ways can 5 seats in a row be occupied if there are 8 people to use them?

11. In how many ways can 6 pupils be seated in 9 seats?

12 In how many ways can 5 boys be seated in 5 front seats?

13 How many signals can be made with 5 different flags if 3 flags, one above another, are used for each signal?

14 In how many ways can a baseball team be arranged if one player always pitches?

15 In how many ways can a baseball team be arranged if the catcher pitcher and first baseman never change positions?

16 If each arrangement of letters spells a word, how many words can be made using all the letters of the word *HOUSE*?

17 If each arrangement of letters spells a word, how many words can be made using all the letters of the word *SPRING*?

18 Six girls are applicants for four secretarial positions in the same office In how many ways is it possible to fill the positions?

19 How many ways are there of arranging the letters of the word *OCTOBER* when each arrangement begins with the letter *O*?

20 In how many ways can a boy have his program arranged if he has 5 classes and there are 8 periods in his school day?

21 How many different batting orders can a baseball team have if the first three places are fixed?

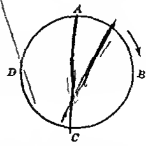
Circular Permutations [A]

Thus far we have studied permutations which can be represented by points on straight lines These permutations are callrd linear permutations We shall now consider circular permutations, which can be represented by points on a circle

A circular permutation is any one of the number of different ways in which a number of different things can be arranged about a circle or any other closed curve

Suppose that we wish to know the number of ways four boys, Allen, Ben, Charles and Dick, can be seated at a round table (We shall designate the boys by A, B C, and D)

The boys can arrange themselves in a line 24 different ways (${}_4P_4$) These arrangements are shown diagrammatically on page 441 Use this diagram as you study the following discussion



Let the boys be seated at the table in any arrangement, such as A, B, C, and D in clockwise order. Because the table is round, we may choose any one of the four positions as the starting point. Starting at A, this arrangement of the boys at the table is $ABCD$, starting at B, their arrangement is $BCDA$, starting at C, their arrangement is $CDAB$, and starting at D, their arrangement is $DABC$. Then for the circular permutation $ABCD$ starting at A, we have the same circular permutation starting at B, the same circular permutation starting at C, and the same circular permutation starting at D. The same is true for any other of the circular permutations starting at A. Then there is only one way to fill the first place in circular permutations of 4 things, instead of the four ways as in linear permutations.

You should note that the circular permutation $ABCD$ in clockwise order is not the same as permutation $ABCD$ in counterclockwise order.

After we have selected one of the boys (A) to fill the first place, there are three ways of selecting the boys for the second place. Then there are two ways to select the boy for the third place. Finally, there is only one way to choose the boy for the fourth place. Then the total number of arrangements of the boys at the table $= 1 \times 3 \times 2 \times 1 = 6$.

In general, if n things are arranged about a circle, each of the n positions may be considered as a starting point, making n arrangements. For each arrangement of the n things on the circle starting at one of these points, there is an identical arrangement starting at each of the other points. Then the first place can be filled in only one way. The remaining $(n-1)$ places can be filled in $(n-1)!$ ways. Then there are $(n-1)!$ ways of arranging n things around a circle (or other closed curve).

We have, therefore, the following rule for finding the number of circular permutations:

The number of circular permutations of n different things is

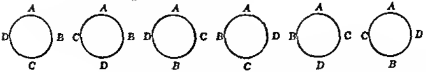
$$\frac{n!}{n}, \text{ or } (n-1)!$$

Example 1 In how many different ways can 8 people be seated at a round table?

Solution $(n-1)! = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Example 2 In how many ways can 4 keys be arranged on a circular key ring?

Solution Without turning the ring over there are $(n-1)!$ arrangements as follows



If we turn the last three rings in the drawing over the fourth arrangement is the same as the first the fifth is the same as the second and the sixth is the same as the third Then the number of permutations is $\frac{(n-1)!}{2} = \frac{3 \times 2}{2} = 3$

EXERCISES

(A)

1 In how many different ways can 7 children join hands to form a circle if they face the center?

2 In how many ways can 8 keys of different kinds be arranged on a ring?

3 In how many different orders can 5 keys be assembled on a key ring?

4 In how many different ways can 6 children be seated at a round table?

5 In how many different ways can 10 adults be seated at a round table?

6 In how many ways can a host place 7 people around a circular table?

7 In how many orders can 3 boys and 3 girls be seated at a round table if the boys and girls alternate? $(3! \cdot 2!)$

8 In how many ways can 5 men and 5 women be seated at a round table if the men and women alternate?

9 How many guests did Luella have if the number of ways of arranging them in a straight line was six times the number of ways of arranging them around a circular table?

Permutations of Things Not All Different[A]

Suppose that we wish to find the number of permutations that can be made with the letters of the word *DELAWARE*, using all the letters in each permutation In this group there are two A's, which are alike, and two E's, which are alike Let us consider any one of the permutations of these letters, such as *DLEERAAW* Neither interchanging the A's nor interchanging the E's in this permutation forms a new permutation If we replace the A's by A_1 and A_2 and replace the E's by E_1 and E_2, the permutation *DLEERAAW* can be changed into the four permutations shown below

$DLE_1E_2RA_1A_2W$ $\qquad$ $DLE_1E_2RA_2A_1W$

$DLE_2E_1RA_1A_2W$ $\qquad$ $DLE_2E_1RA_2A_1W$

Similarly, it can be shown that any other permutation of the letters of *DELAWARE* can be changed into four permutations when the like letters are replaced by unlike letters Since the number of permutations of $DE_1LA_1WA_2RE_2 = 8! = 40{,}320$ the number of permutations of $DELAWARE = \frac{1}{4}(40{,}320) = 10{,}080$

We shall now develop a formula for finding the number of permutations of n things, all taken each time, when r of the things are alike, s things are alike, , and the rest of them are different

If the r like things are replaced by r unlike things, each of the permutations containing the like things can be changed into $r!$ permutations by permuting the r things In like manner, if the s like things are replaced by s unlike things, each of the permutations containing the like things can be changed into $s!$ permutations The same argument holds for other like elements

Now let $N =$ the required number of permutations

$n! =$ the number of permutations of the n things taken n at a time

Then $\quad (r!\,s!\quad)N = n!$ and $N = \dfrac{n!}{r!\,s!\quad}$

The number of permutations of n things taken n at a time with r things alike, s things alike, t things alike, and so on is

$$\frac{n!}{r!\,s!\,t!\cdots}$$

Example How many different arrangements can be made with the letters of the word *MISSISSIPPI*?

Solution $n = 11$, $r = 4$, $s = 4$, and $t = 2$

Then $$N = \frac{11!}{4!\,4!\,2!} = 34\,650$$

There are 34,650 permutations

EXERCISES

ch 15 + 16

[A]

1 In how many ways can the letters of the word *ILLINOIS* be arranged?

2 In how many ways can the letters of the word *COLORADO* be arranged?

3 In how many ways can the letters of the word *LOUISIANA* be arranged?

4 In how many ways can a man give a baseball, bat, and glove to five sons, no son to have more than one gift?

5 In how many ways can 4 pennies, 5 nickels, and 3 dimes be given to 12 children if each child receives a coin?

6 In how many ways can the manager of a baseball team arrange his batting order if the first four positions are fixed?

7 In how many ways can the manager of a baseball team arrange his batting order if four of the players are to head the batting list in any order?

8 In how many ways can 10 flags that are alike, except that 5 are white, 3 are red, and 2 are blue, be arranged one above another?

9 How many arrangements can be made with the letters of the word *PANAMA* if *P* is the first letter of each arrangement?

10 How many arrangements of the letters of the word *VILLAGE* can be made if each arrangement begins with *V* and ends with *E*?

MISCELLANEOUS EXERCISES

[A]

1 How many permutations can be made with the letters a, b, c, d, e, and f, using four letters at a time?

2 In how many ways can 11 books be arranged on a shelf if 3 are of one kind, 3 are of another kind, and the remainder are of another kind?

3. How many permutations can be made with the letters a, b, c, d, and e, using 3 at a time and having d the middle letter in each permutation?

4. How many permutations can be made by using all the letters of the word *TIMES* if each permutation begins and ends with a consonant?

5. In how many of the permutations made by using all the letters of the word *TIMES* is *M* the middle letter?

6. In how many ways can 9 people be seated at a round table?

7. Five people are seated in a row of 7 seats. How many different seating arrangements can be made?

8. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, find how many numbers of 3 digits each can be formed if no digit is repeated in a number.

9. How many license plates having 5 letters each can be made with the English alphabet if no letter is used twice on any plate?

10. How many numbers between 1000 and 2000 can be made with the digits 1, 2, 5, 7, and 8 if no digit occurs twice in any number?

Combinations [A]

A combination is a group of objects in which the arrangement, or order, is not considered. The permutations *AB* and *BA* make the same combination. The six permutations *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, and *CBA* are different permutations, but they are like combinations.

The nine members of a baseball team form one combination of 9 elements, but a very large number of permutations of the players can be made.

We shall now develop a formula for finding the number of combinations of n things taken r at a time.

Let ${}_nC_r =$ the number of combinations of n things taken r at a time.

Let ${}_nP_r =$ the number of permutations of n things taken r at a time.

From each combination of r things there can be formed $r!$ permutations.

Then $$r!\,{}_nC_r = {}_nP_r$$

D,! $${}_nC_r = \frac{{}_nP_r}{r!}$$

The number of combinations of n things taken r at a time is given by the formula

$${}_nC_r = \frac{{}_nP_r}{r!}$$

If r in the formula above is a large number, the formula is unwieldy So we shall develop another formula for finding ${}_nC_r$, as shown below

We shall now prove that ${}_nC_r = {}_nC_{n-r}$

Since $${}_nP_r = n(n-1)(n-2) \quad (n-r+1)$$

$${}_nC_r = \frac{n(n-1)(n-2) \quad (n-r+1)}{r!} \tag{1}$$

Replacing r in this formula by $n-r$, we get

$${}_nC_{n-r} = \frac{n(n-1)(n-2) \quad (r+1)}{(n-r)!} \tag{2}$$

We must show that the right members of equations (1) and (2) are equal

Multiplying both numerator and denominator of the fraction in equation (1) by $(n-r)!$, we get

$${}_nC_r = \frac{n!}{r!(n-r)!} \tag{3}$$

Multiplying both numerator and denominator of the fraction in equation (2) by $r!$, we get

$${}_nC_{n-r} = \frac{n!}{r!(n-r)!} \tag{4}$$

From equations (3) and (4) it follows that

The number of combinations of n things taken r at a time is the same as the number of combinations of n things taken $n-r$ at a time, or

$${}_nC_r = {}_nC_{n-r}$$

Example 1 Find the number of committees of 3 students each that can be selected from 10 students

Solution Since the arrangement of the students is not considered, this is a problem of combinations Then

$$\text{the number of committees} = {}_{10}C_3 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$$

Note that the numerator and denominator have 3 factors each

Example 2. In how many ways can a team of 5 boys be selected from a group of 7 boys?

Solution 1 ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = 21$

Solution 2 We shall now use the formula ${}_nC_r = {}_nC_{n-r}$ to solve this problem $n = 7$, $r = 5$, and $n - r = 2$

$${}_7C_5 = {}_7C_2 = \frac{7 \times 6}{1 \times 2} = 21$$

[A] **EXERCISES**

Find

1 ${}_{10}C_2$ 2 ${}_8C_3$ 3 ${}_{10}C_4$ 4 ${}_8C_6$

5 Show that ${}_{10}C_{10} = 1$

6 In how many ways can a committee of 5 pupils be selected from a group of 30 pupils?

7. How many different groups of 50 each can be made from a group of 52?

8 In Granville County 5 men are to be elected school commissioners If 8 men are candidates, how many different boards of school commissioners are possible?

9. How many straight lines can be drawn through 7 points if no three of the points lie on any straight line?

10 How many straight lines can be drawn through 20 points no three of which lie on any straight line?

11 There are 8 points in space, and no four of them lie in the same plane If any three of these points determine a plane, how many different planes are determined by the 8 points?

12 How many planes are determined by 12 points if no four of the points lie in any one plane?

13 How many diagonals can be drawn in a pentagon?

14 How many diagonals can be drawn in a polygon of 17 sides?

15 In how many ways can a box holding 6 books be filled from a collection of 10 books?

16 How many committees of 4 men can be formed from 10 men?

17 There are 10 teams in a football conference In how many ways can they be paired for games?

18 There are 8 teams in the Alpha Baseball League How many games will be played in the league if each team plays each of the others 5 times?

19 Each major league team is scheduled for 154 games If all the scheduled games are played find the total number of games played by the 8 teams

20 How many different hands of 13 cards each can be dealt from a deck of 52 cards?

21 How many addition problems having two addends of one digit each can be made from the numbers 1, 2, 3 , 9 if no two addends are alike in any addition?

22 Find the number of combinations of 100 things taken 100 at a time

The Total Number of Combinations [A]

The total number of combinations of n things means the total number of groups that can be formed from the n things taking the things one at a time two at a time three at a time, and so on to include n at a time Then

the total number of combinations $= {}_nC_1 + {}_nC_2 +$

The binomial formula is

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} +$$

Let $a = b = 1$ in the formula

Then $\quad 2^n = 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \quad + 1$

and $\quad 2^n - 1 = n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \quad + 1$

Let us examine the terms of the right member of this equation The first term, n, is equal to ${}_nC_1$, the second term, $\frac{n(n-1)}{2!}$, is equal to ${}_nC_2$, the third term is equal to ${}_nC_3$, the rth term is equal to ${}_nC_r$, and the last term, 1, is equal to ${}_nC_n$ Then

$$2^n - 1 = {}_nC_1 + {}_nC_2 + {}_nC_3 + \quad + {}_nC_n$$

The total number of combinations of n things taken 1, 2, 3, things at a time is $2^n - 1$ This fact may be written

$$\sum_{r=1}^{n} {}_nC_r = 2^n - 1$$

Example. In how many ways can 6 balls be drawn from a bag?

Solution $\quad N = 2^n - 1 = 2^6 - 1 = 64 - 1 = 63$

EXERCISES

[A]

1 A boy wishes to invite 10 of his friends to his home In how many ways can he have them as guests?

2 Frank has 20 pennies in his purse In how many ways can he take them from his purse?

3 In how many different ways can 8 letters be placed in a mailbox?

4 How many sums of money can be formed using a cent, a nickel, a dime, and a quarter, one or more at a time?

5 A boys' club has 6 members In how many ways can a committee be formed in the club?

6 A girls' club contains 9 members In how many ways can the president name a committee if the president is not eligible to be a member of it?

Miscellaneous Problems Involving Permutations and Combinations[B]

A grouping problem may be one involving permutations, combinations, or both permutations and combinations When attempting the solution of a grouping problem, you should first determine

whether the groups have arrangement If there is arrangement in the groups, you have a permutation problem, and if the groups do not have arrangement, you have a problem of combinations

When solving a problem involving permutations, you should note whether any elements are alike, and whether repetitions are allowed

Some problems involve both permutations and combinations You should watch for such problems You should never forget the fundamental principle given on page 439

Example 1 In how many ways can 3 red balls and 4 blue balls be arranged in a straight line?

Solution This is a problem in which the groups have arrangement and in which there are like things

$$N = \frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3} = 35$$

Example 2 There are 12 boys and 8 girls in a mathematics club In how many ways can a committee of 3 boys and 2 girls be formed?

Solution There are ${}_{12}C_3$ sets of boys and ${}_8C_2$ sets of girls Each set of the boys can be combined with each set of the girls Then the number of committees that can be formed is ${}_{12}C_3 \times {}_8C_2$, or 6160

EXERCISES

[8]

1 How many different committees consisting of 2 boys and 3 girls can be formed from a class of 15 boys and 18 girls?

2 How many different committees consisting of 3 men and 4 women can be formed from a class of 14 men and 10 women?

3 From 5 red balls and 7 white ones, in how many ways can 6 balls be selected of which 4 are white and 2 are red?

4 From 6 white balls and 4 black ones, in how many ways 90 can 4 balls be selected of which half are white and half are black?

5 A house contains 7 rooms In how many ways can it 210 be decorated if two of the rooms are to be rose color, three of them green, and the others blue?

6 How many positive integers of 3 digits each can be 60 formed with the digits 1, 2, 3, 4, and 5 if all the digits of each number are different?

7. How many positive integers can be formed with the digits 1, 3, 5, 7, and 9 if each integer consists of 4 digits and no digit occurs twice in any integer?

8. In how many ways can 2 women and 3 men be chosen from 5 women and 7 men?

9 A basketball squad consists of 10 players The coach can use 2 of the players as center, 4 of them as guards, and 4 of them as forwards How many different teams can he form from the squad?

10 A high-school baseball coach has 3 pitchers and 2 catchers, and one player for each of the remaining positions on the team How many different teams can the coach place on the diamond?

11 An algebra test consists of 2 parts, each part consisting of 5 problems A student is to select 3 problems from each group In how many ways can a student select the 6 problems he is to solve?

12 A girl wishes to entertain 5 boys and 4 girls In how many ways can she select the guests if she makes her selection from 8 boys and 8 girls?

13. How many integers having two digits each can be formed with the digits 1 to 9 inclusive if no digit is repeated in any integer?

14 How many combinations are there in the multiplication table from 1 times 1 to 9 times 9 inclusive?

15 How many integers of three digits each can be written with the digits 0 to 8 inclusive if no digit occurs twice in any integer? (The number 038 has two digits)

16. A die is a cube whose faces contain 1, 2, 3, 4, 5, and 6 dots, respectively The plural of *die* is *dice* If two dice are thrown, in how many ways can two aces (one dot) turn up? In how many ways can the dice land so that the sum of the dots on the upturned faces is 7?

17. From 8 white and 6 red balls, in how many ways can 3 balls of one color be selected?

18 An urn contains 5 black balls and 6 white balls Five balls are to be drawn from the urn In how many ways can this be done (a) if 2 of the balls are to be black? (b) if 2 or more of the balls are to be black?

19 How many arrangements of letters, each arrangement to consist of 3 vowels and 3 consonants, can be made from 5 vowels and 6 consonants?

NOTE First find the number of combinations and then the number of permutations

Probability [A]

Suppose that a box contains 3 black and 7 white balls If the box is shaken and a person with eyes covered withdraws one ball from the box, this person has 3 favorable chances out of 10 of drawing a black ball The probability of this person's drawing a black ball is $\frac{3}{10}$ The probability of his drawing a white ball is $\frac{7}{10}$

Suppose that you have 7 nickels and 5 dimes in your purse and that you withdraw 2 coins from it What is the probability (chance) that one of the two coins will be a nickel and the other a dime? The total number of combinations of 12 things taken 2 at a time is $\frac{12 \times 11}{1 \times 2}$, or 66 The total number of ways of combining one nickel with one dime is 7×5, or 35 Then there are 35 favorable cases out of a total of 66 cases The probability of having one nickel and one dime in one draw is $\frac{35}{66}$

Since there are 35 favorable cases, there are 31 unfavorable cases The probability of the event's not happening is $\frac{31}{66}$

The probability (or chance) of an event is the ratio of the number of favorable cases to the sum of the favorable and unfavorable cases.

If p denotes the probability that an event will happen, f the number of favorable cases, and u the number of unfavorable cases, then $p = \frac{f}{f+u}$ The probability that the event will fail is $q = \frac{u}{f+u}$. If an event is certain to happen, $u = 0$ and $p = \frac{f}{f+0} = 1$. That is, if

an event is certain to happen, its probability is 1 If an event is certain to fail, its probability is zero The probability of any other case is greater than zero and less than 1

Since $p = \frac{f}{f+u}$ and $q = \frac{u}{f+u}$, $p + q = \frac{f+u}{f+u} = 1$ This means that the probability that an event will happen added to the probability that it will fail is equal to 1

For example, if the probability that an event will happen is $\frac{3}{4}$, the probability that it will not happen is $1 - \frac{3}{4}$, or $\frac{1}{4}$

Odds [A]

If a bag contains 4 white balls and 3 black balls and one ball is withdrawn from the bag, the chance of drawing a white ball is $\frac{4}{7}$ and the chance of drawing a black ball is $\frac{3}{7}$ In drawing for a white ball, the number of favorable cases is 4 and the number of unfavorable cases is 3 We say that the odds are in favor of drawing a white ball *In this case the odds are 4 to 3 in favor of drawing* a white ball and 4 to 3 against drawing a black ball

If the number of favorable cases for any event is greater than the number of unfavorable cases, the odds are in favor of the event, and if the number of favorable cases is less than the number of unfavorable cases, the odds are against the event

Compound Probability [A]

If the probability of a first event is p_1 and after this event has happened the probability of a second event is p_2, the probability that both events will occur in the order stated is $p_1 p_2$

Example 1 A bag contains 3 white balls and 4 black ones What is the chance in two drawings of drawing 2 white balls from the bag?

Solution 1. The chance of drawing a white ball in one draw is $\frac{3}{7}$ After a white ball is withdrawn, there are 2 white balls and 4 black balls left Then the chance of drawing a white ball is $\frac{2}{6}$, or $\frac{1}{3}$ Then the chance of drawing 2 white balls is $\frac{3}{7} \times \frac{1}{3}$, or $\frac{1}{7}$

Solution 2

$$f + u = {}_7C_2 = 21$$
$$f = {}_3C_2 = 3$$
$$\frac{f}{f+u} = \frac{3}{21} = \frac{1}{7}$$

Example 2 If three coins are tossed in the air simultane ously, what is the chance that only one will turn up heads What is the probability of tossing three heads?

Solution. There are 8 possible cases, for if we name the coins A B, and C, the possible cases are

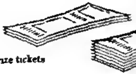
A head, B head, C head	A tail, B head, C head
A head, B head, C tail	A tail, B head, C tail
A head, B tail, C head	A tail, B tail, C head
A head, B tail, C tail	A tail, B tail, C tail

One head and two tails can result 3 times Then the probabil ity that one head will be thrown is $\frac{3}{8}$ Since there is only on case of tossing 3 heads, the probability of tossing 3 heads is $\frac{1}{8}$

Example 3 What is the chance of throwing a total of 7 with two dice?

Solution. Each die has 6 faces, and any face of one may turn u with any face of the other Then there are 6×6, or 36, pos sible cases The total of 7 may be made with faces reading and 4, 6 and 1, and 5 and 2 There are two ways in whicl each of these may occur Then there are 6 favorable cases The chance of making a total of 7 is $\frac{6}{36}$, or $\frac{1}{6}$

Example 4 A lottery sells 10 tickets and offers 3 prizes What is the chance of winning one or more prizes with four tickets?



3 prize tickets

7 blank tickets

Solution 1 Think of the prize tickets being in one pile and the blank tickets in another pile The favorable cases are

3 prize tickets and 1 blank ticket
2 prize tickets and 2 blank tickets
1 prize ticket and 3 blank tickets

3 prize tickets and 1 blank ticket can be drawn in ${}_3C_3 \cdot {}_7C_1 = 1 \cdot 7$ $= 7$ ways

2 prize tickets and 2 blank tickets can be drawn in ${}_3C_2 \cdot {}_7C_2$ $= 3 \times 21 = 63$ ways

1 prize ticket and 3 blank tickets can be drawn in ${}_3C_1 \cdot {}_7C_3 = 3 \cdot 35 = 105$ ways

The total number of ways of drawing 4 tickets from 10 tickets $= {}_{10}C_4 = 210$

Then the chance of drawing one or more prizes is $\frac{175}{210}$, or $\frac{5}{6}$

Solution 2 Think of all the tickets being in one pile, which consists of 3 prize tickets and 7 blank tickets Remember to try not to win a prize The chance of not drawing a prize ticket in the first draw is $\frac{7}{10}$ If no prize is drawn in the first draw, there are 6 blank tickets left in the pile Then the chance of not drawing a prize ticket in the second draw is $\frac{6}{9}$ If no prize ticket has been drawn, the chance of drawing a prize in the third draw is $\frac{5}{8}$ If no prize ticket has been drawn by now, the chance of drawing a prize ticket in the fourth draw is $\frac{4}{7}$ Then the chance of not drawing a prize ticket in 4 draws $= \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{1}{6}$

Then the chance of winning one or more prizes $= \frac{6}{6} - \frac{1}{6} = \frac{5}{6}$

Example 5 There are 13 spades in a deck of cards If 3 cards are drawn from the 13 spades, what is the chance of drawing a king and queen?

Solution There are ${}_{13}C_3$, or 286, combinations of three cards The king and queen can be drawn with each of the remaining 11 cards Then the chance of drawing a king and queen is $\frac{11}{286}$, or $\frac{1}{26}$

Example 6. The probability of one boy's solving a problem is $\frac{3}{4}$, and the probability of another boy's solving it is $\frac{2}{3}$

a. What is the probability of both boys' solving the problem?

b What is the probability that neither boy will solve it?

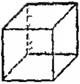
c What is the probability that one or both boys will solve it?

d. What is the probability of one boy's not solving it?

e. What is the probability that one boy will and the other boy will not solve it?

Solution The probability $\frac{3}{4}$ can be represented by a solid having 4 equal faces, of which 3 may be marked "right" and one marked "wrong" Since $\frac{2}{3} = \frac{4}{6}$, the probability $\frac{2}{3}$ can be represented by the faces of a cube, of which 4 may be marked "right" and 2 marked "wrong" (See the diagrams on the following page)

3 right faces
1 wrong face



4 right faces
2 wrong faces

If the solids are tossed into the air, the chance of a right face landing on the bottom is $\frac{3}{4}$ for the first figure and $\frac{4}{6}$ for the second figure. The total number of combinations is 4×6, or 24. These are distributed as follows:

FIRST SOLID		SECOND SOLID	CASES
3 right	with	4 right	12
1 wrong	with	2 wrong	2
3 right	with	2 wrong	6
1 wrong	with	4 right	4
		Total	24

a. For both boys solving the problem, the number of favorable cases is 12, and the probability of both of them solving the problem is $\frac{12}{24}$, or $\frac{1}{2}$. Also, $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$.

b. For both boys not solving the problem, the number of favorable cases is 2, and the probability of both not solving it is $\frac{2}{24}$, or $\frac{1}{12}$.

c. The number of favorable cases of one or both boys solving the problem is $12 + 6 + 4$, or 22, and the probability of one or both solving it is $\frac{22}{24}$, or $\frac{11}{12}$. Also, $1 - \frac{1}{12} = \frac{11}{12}$.

d. The number of favorable cases of one boy not solving the problem is $2 + 6 + 4$, or 12, and the probability of one boy not solving the problem is $\frac{12}{24}$, or $\frac{1}{2}$. Also, $1 - \frac{1}{2} = \frac{1}{2}$.

e. The number of favorable cases of one boy solving the problem and one not solving it is $6 + 4$, or 10, and the probability of one solving it and one not solving it is $\frac{10}{24}$ or $\frac{5}{12}$. Also, $\frac{11}{12} - \frac{1}{2} = \frac{5}{12}$.

EXERCISES

[A]

1. There are 4 white balls and 10 black ones in a bag. What is the chance of drawing a white ball in one draw? What is the chance of getting a black ball in one draw?

2 What is the probability of drawing an ace of spades when drawing one card from a deck of 52 cards?

3 A boy tosses two coins

a. What is the chance of tossing 1 head?

b What is the chance of tossing 2 heads?

c What is the chance of tossing 1 head and 1 tail?

4 Four cards are drawn from a suit of 13 cards What is the probability that neither the king nor the queen will be drawn? (There are ${}_{13}C_4$ total cases and ${}_{11}C_4$ favorable cases You may follow the plan of Solution 2 of Example 4)

5 An urn contains 6 black balls, 4 white balls, and 2 red balls If 2 balls are drawn from the urn, what is the probability of both balls being white?

6 A deck of cards contains 13 hearts What is the probability of getting a heart if 1 card is drawn? of getting 3 hearts if 3 cards are drawn?

7. A deck of cards containing 52 cards is separated into suits of 13 cards each If a card is drawn from each suit, what is the probability that all four cards are aces?

8 If the probability of an event is $\frac{3}{5}$, what are the odds in favor of the event?

9. If there are 30 pupils in your class and 4 are drawn by lot, what are the odds against your being drawn?

10 The probability of an event's happening is $\frac{17}{20}$ What is the probability of its not happening?

11 If the chance that an event will happen is $\frac{3}{7}$, what are the odds against its happening?

12. When two cards are drawn from a deck, what is the chance that (a) both are aces? (b) one or both are aces?

13. If one million people have equal chances of being called by telephone from a broadcasting show, what is the chance of (a) one person's receiving a call? (b) one person's receiving two consecutive calls?

14 If you are in an infantry company of 240 men and 10 of them are drawn by lot for furlough, what is the chance of one of the men having a furlough with you?

15 An algebra class consists of 30 pupils who are seated in 6 rows of 5 pupils each What is the probability that 2 particular pupils will occupy 2 of the front seats?

16 A bag contains 7 white and 10 black buttons If 3 buttons are taken from the bag what is the chance of getting 1 black button and 2 white buttons? of getting 2 black buttons and one white button?

17 What is the chance of tossing a total of 9 in a single throw of 2 dice?

18 Of 100 000 persons at the age of 10 years only 95,965 will live to be 16 years of age and only 69,517 will live to be 50 years old If you are 16 years old what is your chance of living to be 50 years old?

[B]

19 What is the chance of drawing an ace of spades and a king of hearts in 2 draws from a deck of 52 cards?

20 In a lottery 40 tickets are to be sold and 2 prizes are to be given If you buy 4 tickets what is your chance of winning a prize?

21 If 4 cards are drawn from a deck of 52 cards what is the chance of getting 3 kings? (Place the 4 kings in one pile)

22 *A* s chance of solving a problem is $\frac{1}{2}$ and *B* s chance of solving it is $\frac{2}{3}$

a What is the chance of both solving it?

b What is the chance of either solving it?

c What is the chance of neither solving it?

23 What is the chance of tossing a total of 11 with two throws of a die?

24 A deck of cards is separated into suits, and a card is drawn from each suit What is the chance of getting an ace, a king a queen, and a jack on this draw?

25 Assuming that $\frac{1}{2}$ is the probability that any child born is a boy, what is the probability that at least three children in a family of six offspring are girls?

Checking Your Understanding of Chapter 17

Before you begin the chapter review be sure that

Should you review?

1 You know the meaning of a permutation (p 439), a combination (p 439), the odds in favor and the odds against the happening of an event (p 457)

2 You remember and can apply the following formulas

$${}_nP_r = n(n-1) \quad (n-r+1) \qquad \text{(p 442)}$$

$${}_nP_n = n! \qquad \text{(p 442)}$$

$$N = (n-1)! \qquad \text{(p 445)}$$

$$N = \frac{n!}{r!\,s!} \qquad \text{(p 447)}$$

$${}_nC_r = \frac{{}_nP_r}{r!} \qquad \text{(p 450)}$$

3 You can solve problems in probability (p 460)

4 You can spell and use correctly the words

combination (p 439)	odds (p 457)
permutation (p 439)	probability (p 456)

CHAPTER REVIEW

(A)

1 What is the difference between the number of permutations of 10 things taken 4 at a time and the number of combinations of 10 things taken 4 at a time?

2 How many signals can be made with 8 whistles by blowing 3 of them in succession?

3 How many different combinations can be formed with 18 letters, using 10 at a time?

4 How many permutations can be formed from the 26 letters of the alphabet, using 6 at a time?

5 From a group of 40 boys, how many different committees of 5 boys each can be formed?

6 Find the number of combinations that can be formed from 20 things, taking 8 at a time

7. There are 14 points in a plane and no 3 of them lie on any straight line How many different triangles having these points as vertices can be formed?

8 If the odds are 3 to 4 against an event happening, what is the probability of its happening?

9 How many different positive integers of 4 different digits each can be formed with the digits 1 to 9 inclusive?

10 How many different words each consisting of 3 consonants and 2 vowels can be formed with 7 consonants and 3 vowels, assuming that each arrangement forms a word?

NOTE Find the number of combinations that can be made of 7 consonants taken 3 at a time and the number of combinations of 3 vowels taken 2 at a time Next find the number of combinations of consonants and vowels Each of these combinations has 5! permutations

11 How many different words each having 2 consonants and 2 vowels can be made from 10 consonants and 5 vowels?

12 If 2 dice are tossed, what is the probability that both will turn 5's?

13 If a die is thrown 4 times in succession, what is the chance that a 2 is not thrown?

14 In how many ways can 7 men be seated at a round table?

15 If a die is thrown 3 times in succession what is the probability that a 4 will be thrown each time?

16 A lottery is selling 25 tickets and offering 2 prizes What is the probability of drawing at least one prize with 4 tickets?

17 From a group of 8 men and 12 women a committee of 5 is to be appointed What is the probability of 4 men's being on the committee?

CHAPTER TEST

[A]

1 Find the number of permutations of 6 things taken 2 at a time

2 Find the number of combinations of 20 things taken 4 at a time

3 There are 12 girls applying for the positions of typist and clerk In how many ways can the positions be filled if each of the girls can fill either position?

4 In how many ways can 8 boys be arranged in a straight line?

5. If the probability of an event's happening is $\frac{9}{13}$, what is the probability of its not happening?

6 A high-school baseball league consists of 8 teams In how many ways can they be paired for games?

7 In how many ways can 10 people be seated around a circular table?

8. If 2 dice are thrown, what is the chance of both being 5's?

9 In how many ways can the cards of a deck of 52 cards be dealt to 4 players?

10 If 6 cards are drawn from a deck of 52 cards, what is the chance of all of them being spades?

CHAPTER 18

Statistics

In this chapter you will learn
something about the way
a statistician thinks and works

Many of the achievements of our present civilization have become possible because we have learned to rely on numerical information instead of personal opinion and unconfirmed reports A great shoe manufacturing company, for example, could not hope to stay in business if it had to depend upon guesswork concerning the proportion of its possible customers who wear each of the various shoe sizes It must know Insurance companies must know the proportion of people who live to each of the different ages Government and social agencies must have numerical information about people and salaries and crops and manufacturing output

Meaning of Statistics [A]

Statistics is the science of collecting, organizing, and interpreting numerical facts It helps us to recognize and summarize concisely the characteristics of groups (perhaps groups of people, or test scores, or prices) Sometimes it helps us to recognize trends and their possible consequences Sometimes it helps us to express mathematically such relationships as those between weather and crop production Sometimes it helps us to judge the quality of the whole by a sample

Gathering Data [A]

In statistics, attention to gathering good data is important because a statistician's conclusions can be no more accurate than the information he uses to form the conclusions

In the brief presentation of statistical methods in this chapter you will not be asked to assemble data That has been done for you Your only task is to learn a few of the procedures a statistician uses in organizing and interpreting data after it has been assembled

Recording Information in Frequency Distributions [A]

Having gathered suitable information, the statistician's next task is to record it in such manner that it can be analyzed effectively For example, in one school in which an algebra class was given a test covering topics which it had just finished studying, the teacher, after checking the papers, recorded the scores in her class book by writing beside each pupil's name the score he had made Such a tabulation enabled her to find instantly the score of each pupil, but it did not enable her to see results for the class as a whole

To study the performance of the class as a unit, the teacher made the following tabulation known as a frequency distribution

Notice that she arranged the scores in rank order (order of size) in one column and listed their frequencies (number of times each score appeared) in another column The tally marks were temporary marks she made as she recorded the scores from the test papers

SCORES MADE ON ALGEBRA TEST

Scores	Tally	Frequency
100	//	2
95	////	4
90	~~////~~	5
85	~~////~~ /	6
80	///	3
75	/	1
		Total 21

This tabulation enabled the teacher to see that the range of scores was from 75 to 100 It showed that the mode (item appearing most frequently) was 85

The distribution helped her to find the arithmetic mean of the scores You know that the arithmetic mean (often called *average*) is the quotient obtained when the sum of all the scores is divided by the number of scores She wrote

Arithmetic Mean =

$$\frac{2(100) + 4(95) + 5(90) + 6(85) + 3(80) + 75}{21} = 88\tfrac{1}{3}$$

The distribution also helped her to find the median (middle score) Since there was a total of 21 scores she knew the median was the 11th score She counted down from the top in the frequency column and located the 11th score in the group of 5 which registered the frequency of the score 90 She knew 90 was the median score

Had there been an even number of scores, this teacher might have taken the score half way between the two middle scores as the median For example, in the distribution at the right she might have taken the score half way between the fifth and sixth scores as the median According to this definition it would have been $92\frac{1}{2}$ Most statisticians, however, *weight* such medians Since 95 appears 3 times and 90 appears 4 times, they take $\frac{3(95) + 4(90)}{7} = 92\frac{1}{7}$ The latter method is considered the more accurate, and is the method we shall use in this book

Score	Frequency
100	2
95	3
90	4
85	1
	Total 10

EXERCISES [A]

Make a frequency distribution for each of the following sets of numbers and find the range, the arithmetic mean, the median, and the mode

1 The temperatures, in degrees, recorded in a midwest town during the last two weeks of one July were

100, 101, 102, 100, 101, 99, 99, 98, 100, 100, 99, 75, 74, 82

2 The prices of twelve homes sold by the Sunny Acres Real Estate Company last month were

\$10,000, \$18,000, \$15,000, \$20,500, \$10,000, \$30,000, \$15,000, \$10,000, \$18,000, \$10,000, \$15,000, \$8,000

3 The tax in cents, on a gallon of gasoline, charged by each of the states and the District of Columbia is

6, 5, 6 5, 6, 6, 4, 5, 7, 6, 6, 5, 4, 5, 5, 7, 7, 6, 6, 5, 4 5, 5, 7, 3, 6, 6, 4 5, 5, 3, 6, 4, 7, 5, 5, 6 5, 6, 5, 4, 7, 5, 7, 4, 5, 5, 6, 6 5, 5, 4, 5, 5

4 The highest temperatures, in degrees, recorded for New York City for each day of one December were

45, 49, 58, 59, 59, 62, 62, 54, 60, 61, 48, 49, 48, 54, 47, 39, 28, 26, 31, 44, 47, 48, 49, 30, 42, 43, 42, 44, 45, 48, 40

Making Frequency Distributions by Classes [A]

When the number of items under consideration is large, or involves many classifications, it is usually advisable to group the items For example, consider the following scores made by 50 contestants in a state algebra contest

145, 138, 132, 131, 127, 126, 125, 125, 124, 121, 121, 120, 120, 118, 117, 117, 116, 116, 115, 115, 115, 115, 113, 113, 113, 111, 111, 110, 110, 110, 109, 109, 109, 109, 108, 106, 106, 103, 103, 102, 101, 100, 98, 98, 96, 90, 87, 85, 81, 75

Since making a distribution of the individual scores would involve 31 classifications, it is probably better to group the items In this distribution it seems sensible to group the items into classes (groups) of ten by putting all the scores in the eighties into one group, all the scores in the nineties into another group, and so on

Since the original scores are given accurate to the nearest unit we shall make the break between classes at the point half-way between two units, thus

$$\underbrace{\ldots\ 78\quad 79}_{69.5-79.5}\overset{79.5}{\ } \underbrace{80\quad 81\quad 82\quad 83\quad \ldots\quad 87\quad 88\quad 89}_{79.5-89.5}\overset{89.5}{\ } \underbrace{90\quad 91\quad 92\ \ldots}_{89.5-99.5}$$

This means that the class units will be 69.5–79.5, 79.5–89.5, and so on. Our distribution will appear as follows

FREQUENCY DISTRIBUTION FOR ALGEBRA CONTEST SCORES

Class	m	f	fm	Cumulative Frequency
139.5–149.5	144.5	1	144.5	1
129.5–139.5	134.5	3	403.5	4
119.5–129.5	124.5	9	1120.5	13
109.5–119.5	114.5	17	1946.5	30
99.5–109.5	104.5	12	1254.0	42
89.5–99.5	94.5	4	378.0	46
79.5–89.5	84.5	3	253.5	49
69.5–79.5	74.5	1	74.5	50
			$\Sigma fm = 5575.0$	

When we arrange data in classes we must choose one number to *represent* the value of the numbers within each class. We have used the midpoint of each class as the representative of the class, and have written the values in the column headed m, which stands for midpoint. The frequency column (f) shows the frequency within each class, and the fm column shows the *product* of the frequency and the class value. The fm column makes it easy to find the arithmetic mean of the class values. Do you see that if we divide 5575.0, the sum of the terms in the fm column, by 50, we get the arithmetic mean of the class values—111.5? In the table, Σfm means "the sum of the fm terms." The symbol Σ (pronounced sigma) is a Greek letter often used to denote "the sum of."

For scores arranged by classes we use the arithmetic mean of the class values as the arithmetic mean of the scores. Obviously, however, this arithmetic mean is not identical with the arithmetic mean of the actual scores because when scores are grouped, the individual scores lose their identity and are represented by their class values. The arithmetic mean of the actual scores above is 111.3

[A] EXERCISES

1 The following numbers are the percentages of households, by states and the District of Columbia, which had telephones in a recent year

64, 68, 64, 80, 71, 89, 78, 79, 73, 80, 70, 71, 80, 72, 79, 81, 66, 59, 63, 76, 76, 78, 68, 90, 53, 46, 40, 35, 45, 49, 43, 51, 39, 29, 33, 51, 61, 57, 60, 60, 62, 75, 42, 49, 74, 57, 71, 64, 73

Using classes of 10, make a frequency distribution of the percentages, then find the arithmetic mean, the mode and the median

FREQUENCY DISTRIBUTION OF THE NUMBER OF TELEPHONES IN THE VARIOUS STATES

Class	m	f	fm	Cumulative Frequency
99 5-89 5				
89 5-79 5				
79 5- ?				

2 The following numbers represent the death rates, by states and the District of Columbia, per 100,000,000 miles of motor vehicle travel in a recent year

9 5, 9 8, 8 3, 6 6, 5 8, 3 6, 7 5, 2 5, 7 5, 7 9, 7 5, 7 2, 7 4 5 7, 7 0, 10 4, 9 1, 5 3, 6 5, 4 0, 7 4, 5 9, 7 5, 6 4, 8 4, 5 9, 10 3, 4 9, 4 1, 11 5, 6 0, 8 2, 7 6, 6 5, 6 3, 6 0, 5 1, 2 8, 11 0, 6 5, 8 3, 6 6, 7 3, 5 8, 7 5, 5 1, 8 7, 7 1, 9 1

Using classes of 1 95–2 95, 2 95–3 95, and so on, make a frequency distribution and find the arithmetic mean, the median, and the mode

3. The weights of the boys reporting for try-outs for a freshman basketball team were

118, 142, 105, 157, 160, 152, 138 112, 113, 127, 160, 95, 73, 112, 103, 107, 124, 171, 107, 121, 133, 104, 117, 135

Using classes of ten, make a frequency distribution for the weights, then find the arithmetic mean, the median, and the mode for the distribution

Graphical Representation [A]

In statistical work we frequently make use of graphs A graph with vertical bars representing the frequencies of the various classes along the horizontal axis is called a histogram In a histogram usually no space is left between the bars The histogram at the right shows the scores from the table on page 468 Notice that the scores 70, 80, and so on are written a tiny bit to the right of the edge of a bar That is because the classes are separated at the points 69 5, 79 5, and so on The histogram shows that a score of 70 falls within the first group a score of 80 within the second, and so on

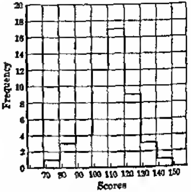


Fig 1 Histogram of scores made in a state algebra contest

Sometimes a line graph called a frequency polygon is used In a frequency polygon lines are drawn connecting the mid points of the class intervals A frequency polygon of the scores in the histogram above is shown in Fig 2 The frequency polygon has been drawn on top of the histogram to show how it connects the midpoints of the class intervals

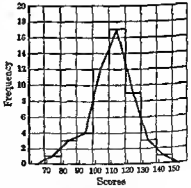


Fig 2 Frequency polygon of scores made on state algebra contest

Notice that both the histogram and the frequency polygon show a slight bell shape Had the test been given to thousands of algebra pupils of similar ability instead of to fifty, the chances are that the scores would form an almost perfect bell shaped curve, as shown in Fig 3 on the next page This curve is known as the normal probability curve

The Meaning of Normal Probability [B]

You learned on page 456 that the probability of an event is the ratio of the number of favorable cases to the sum of the favorable and unfavorable cases. For example, if you toss two coins A and B, they may fall in 2^2 or 4, ways. Of these there is the possibility of A head, B head; A head, B tail; A tail, B head; and A tail, B tail. In other words, of the 4 possible ways that the coins may fall, you have 1 chance in 4 to get 2 heads, 2 chances in 4 to get 1 head, and 1 chance in 4 to get 0 heads.

Fig 3 Normal probability curve

The probabilities of turning up heads on tosses of a coin can be expressed by the terms of a binomial expansion. To see how this is possible we need first to prove the following theorem.

Theorem If p is the probability that an event will happen and $q = 1 - p$ is the probability that it will not happen in any single event, the probability that the event will happen exactly r times in n trials is ${}_nC_r p^r q^{n-r}$.

PROOF The r trials can be selected from the n trials in ${}_nC_r$ ways. The probability that the event will happen r times and fail in the remaining $n - r$ times is $p^r q^{n-r}$. Then the probability of getting exactly r successes in n trials is ${}_nC_r p^r q^{n-r}$.

Now let us expand $(p + q)^n$, obtaining

$$(p+q)^n = p^n + \frac{n}{1} p^{n-1} q^1 + \frac{n(n-1)}{2!} p^{n-2} q^2 + \frac{n(n-1)(n-2)}{3!} p^{n-3} q^3 + \cdots + q^n \qquad (1)$$

Remembering that ${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$, we can write (1) using the notation ${}_nC_r$ for the coefficients, and obtain

$$(p+q)^n = {}_nC_0 p^n + {}_nC_1 p^{n-1} q^1 + {}_nC_2 p^{n-2} q^2 + {}_nC_3 p^{n-3} q^3 + \cdots + {}_nC_n q^n \qquad (2)$$

By the theorem just proved, we see that the successive terms in the right member of (2) are the probabilities that the event will happen

exactly n times, exactly $n-1$ times, exactly $n-2$ times, and so on to exactly 1 time, and finally to exactly zero times. We see, therefore, that if n trials are made of an event in which the probability of success is p and the probability of failure is q, the probabilities of n, $n-1$, $n-2$, ..., 0 successes are given by the successive terms of the expansion of $(p+q)^n$.

If we consider the case of tossing a coin n times, we have $p=\frac{1}{2}$ and $q=\frac{1}{2}$ since a coin has one chance in two to fall with head up. Equation (1) then becomes

$$\left(\frac{1}{2}+\frac{1}{2}\right)^n = \frac{(1+1)^n}{2^n} = \frac{1}{2^n} + \frac{n}{2^n} + \frac{n(n-1)}{2!\ 2^n} + \frac{n(n-1)(n-2)}{3!\ 2^n} + \cdots + \frac{1}{2^n}. \tag{3}$$

The successive terms of (3) give the probabilities of obtaining n, $n-1$, $n-2$, ..., 0 heads in n tosses of a coin.

Example If we toss 3 coins, what is the probability of obtaining 3 heads, 2 heads, 1 head, or 0 heads?

Solution Since tossing 3 coins is equivalent to tossing 1 coin 3 times, we have $n=3$. Thus (3) becomes

$$(\tfrac{1}{2}+\tfrac{1}{2})^3 = \underset{\substack{\text{Probability}\\ \text{of obtaining}\\ \text{3 heads}}}{\tfrac{1}{8}} + \underset{\substack{\text{Probability}\\ \text{of obtaining}\\ \text{2 heads}}}{\tfrac{3}{8}} + \underset{\substack{\text{Probability}\\ \text{of obtaining}\\ \text{1 head}}}{\tfrac{3}{8}} + \underset{\substack{\text{Probability}\\ \text{of obtaining}\\ \text{0 heads}}}{\tfrac{1}{8}}$$

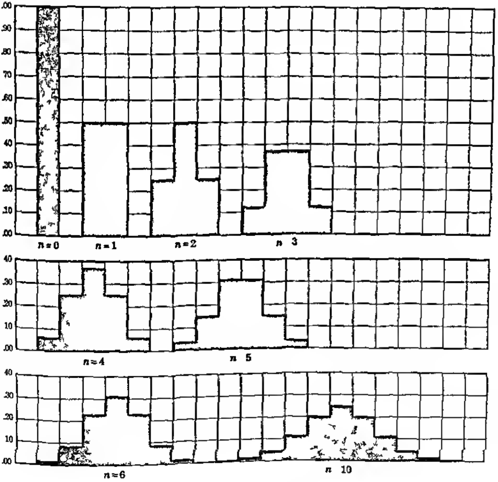
If you have difficulty visualizing the various probabilities just indicated, study the following table which shows the ways in which 3 tossed coins may fall.

	1 case of 3 heads	3 cases of 2 heads			3 cases of 1 head			1 case of 0 heads
1st coin	h	h	h	t	h	t	t	t
2d coin	h	h	t	h	t	h	t	t
3d coin	h	t	h	h	t	t	h	t

Let us now make a table of the probabilities of obtaining n heads, $n-1$ heads, $n-2$ heads, ..., 0 heads when n coins are tossed and n takes on the values 0, 1, 2, 3, 4, ...

n	Binomial Expression		Probabilities of the following numbers of heads 0	1	2	3	4	5	6	7
0	$(\frac{1}{2}+\frac{1}{2})^0$	=	$\frac{1}{1}$							
1	$(\frac{1}{2}+\frac{1}{2})^1$	=	$\frac{1}{2}$	$+\frac{1}{2}$						
2	$(\frac{1}{2}+\frac{1}{2})^2$	=	$\frac{1}{4}$	$+\frac{2}{4}$	$+\frac{1}{4}$					
3	$(\frac{1}{2}+\frac{1}{2})^3$	=	$\frac{1}{8}$	$+\frac{3}{8}$	$+\frac{3}{8}$	$+\frac{1}{8}$				
4	$(\frac{1}{2}+\frac{1}{2})^4$	=	$\frac{1}{16}$	$+\frac{4}{16}$	$+\frac{6}{16}$	$+\frac{4}{16}$	$+\frac{1}{16}$			

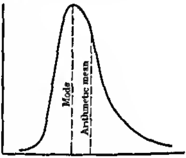
The histograms for these and other values of n are shown below In the histograms the fractions have been expressed as decimals to



make comparison easier Notice that as n increases the histogram approaches a bell-shaped curve The normal distribution curve is the limiting value of the histogram of $(\frac{1}{2}+\frac{1}{2})^n$ as n approaches infinity It can be proved that the equation of the curve when n becomes infinite is $y = e^{-x^2}$, where e has the approximate value 2 718

The normal distribution curve also describes the frequencies in which many other measurable tendencies are likely to appear For example, if you were to make a frequency polygon of the heights of thousands of six-year-old children, or the foot lengths of all the men in the army, or the weights of thousands of apples, you would get a curve which approximates the normal distribution curve

Do not be led into thinking, however, that a distribution which does not fit into the bell-shaped pattern is not normal If, for example, you were to make a frequency polygon of the ages at which people learn to drive automobiles, you would get a curve somewhat like the one at the right Since laws keep most of us from driving before we are sixteen, the curve rises sharply at about that age though it tapers off gradually We say that such a curve is skew (lop-sided) Even though it is not bell-shaped, it is normal for the particular situation which it represents



EXERCISES

(A)

1 Make a histogram of the information in exercise 1, p 471

2 Make a frequency polygon for the information you used in exercise 1

3 Make a histogram based on exercise 3, p 471

(B)

4 Toss 3 coins 24 times and record the heads and tails on each throw Did you get zero heads one-eighth of the time, 1 head three-eighths of the time, 2 heads three-eighths of the time, and three heads one-eighth of the time? Try two or three more sets of 24 tosses and answer the same questions for the total number of tosses Did you come nearer obtaining one-eighth, three-eighths, three eighths, one-eighth?

Averages [A]

Although the arithmetic mean is commonly known as the aver age actually the arithmetic mean the median and the mode are all averages of different kinds In cases where the numbers under consideration form a geometric progression the geometric mean which you studied in Chapter 13 is another average to be considered Each of these averages in its own special way tells something about the group of numbers being studied

Of the averages the arithmetic mean is generally the best repre sentative of a group of numbers but indiscriminate use of the arith metic mean can be misleading For example it would be foolish to assume that because two communities each have the same arithmetic mean income per week the people of one should be as well to do financially as the people of the other Actually though the people in one community might be able to live comfortably most of the people in the other might be very poor You can see how that might be possible by studying the following two distributions

COMMUNITY 1

Income per Family in Dollars	Frequency
95	21
	Total 21

COMMUNITY 2

Income per Family in Dollars	Frequency
1000	1
50	20
	Total 21

$$\text{Arithmetic mean} = \frac{21(95)}{21} = 95$$

$$\text{Arithmetic mean} = \frac{1000 + 20(50)}{21} = 95+$$

Do you see that in Community 2 the one income of \$1000 per week so raises the mean income of the whole community that the arith metic mean no longer represents any of the incomes? In Commu nity 1 however the arithmetic mean describes each income in the group exactly

Measures of Dispersion [A]

We see that while the averages of a group of numbers give us in formation about the group they need to be supplemented with other measures

Let us consider the scores made by Gene H and Bill M in a series of Junior amateur golf tournaments

Gene H	76,	78,	76,	76,	74
Bill M	72,	73,	74,	78,	83

While both boys have an arithmetic mean score of 76, Bill's score fluctuates more than Gene's Bill's scores show a spread of 11 points (4 below the arithmetic mean and 7 above) while Gene's show a spread of only 4 points (2 below and 2 above the arithmetic mean) We say that Gene is a more consistent player than Bill

The degree of spread (or scatter) of numbers from one of their averages is called their dispersion Bill's scores show more dispersion than Gene's

We need a way to measure dispersion While there are three ways in general use, we shall study only one—the standard deviation

In the table below, the standard deviations for the scores of Gene and Bill have been computed To learn the procedure let us examine, in detail, the computation of the standard deviation for Gene's scores

GENE			BILL		
Score	d	d^2	Score	d	d^2
78	+2	+4	83	+7	+49
76	0	0	78	+2	+4
76	0	0	74	−2	+4
76	0	0	73	−3	+9
74	−2	+4	72	−4	+16

Gene: A = 76

$$\Sigma d^2 = +8$$

$$\frac{\Sigma d^2}{n} = \frac{8}{5}$$

$$\text{S D} = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{8}{5}}$$

$$\text{S D} = 1.26$$

Bill: A = 76

$$\Sigma d^2 = 82$$

$$\frac{\Sigma d^2}{n} = \frac{82}{5}$$

$$\text{S D} = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{82}{5}}$$

$$\text{S D} = 4.05$$

We first computed the arithmetic mean (A) for the scores, thus, $\frac{78+76+76+76+74}{5} = 76$ We then computed the deviation (d) from A for each score and wrote it in the d column Since the sum of the deviations of a set of numbers from their arithmetic mean is zero we squared the deviations to eliminate negative numbers We then wrote these squares in the d^2 column and found their sum, + 8

(Remember that the symbol Σ indicates "the sum of") Next we divided Σd^2 by the number of items (n), obtaining $\frac{\Sigma d^2}{n} = \frac{8}{5}$. Finally, we took the square root of each member, obtaining the standard deviation (S D), thus $\sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{8}{5}} = 1\ 26$

Finding the standard deviation for Bill's scores by the same procedure, we have S D $= 4\ 05$ As we expected, the standard deviation for Bill's scores is larger than the standard deviation for Gene's

Rules for Computing Standard Deviation

1. Find the arithmetic mean (A) of the items
2. Find the deviation (d) from A for each item
3. Find d^2 for each item
4. Find Σd^2 and divide it by the number of items (n)
5. The square root of $\frac{\Sigma d^2}{n}$ is the standard deviation of the items

EXERCISES

(A)

In exercises 1–4 find the standard deviations for the sets of numbers given

1 The scores made on a science test were 150, 147, 145, 141, 138, 137, 132, 129, 114

2 The amounts of money Mollie King saved from her salary during each week for seven weeks were \$1 50, \$2 50, \$0 25, \$4 00, \$3 00, \$1 00, \$3 50

3 The numbers of books loaned by a library during each day of one week were 36, 31, 30, 24, 45, 50

4 The measurements recorded by each member of an eighth grade class upon measuring the width of the class room were

17 ft 4 in , 17 ft 3 in , 17 ft 5 in , 17 ft 6 in , 17 ft 6 in ,
17 ft 5 in , 17 ft 7 in , 17 ft 5 in , 17 ft 8 in , 17 ft 6 in

5 The normal monthly temperatures for Miami and Honolulu follow After making the necessary computations, com-

pare the arithmetic mean temperatures of the two cities and the standard deviations of the temperatures

Miami 68, 69, 71, 74, 77, 80, 82, 82, 81, 78, 72, 69
Honolulu 72, 72, 72, 73, 75, 77, 78, 78, 78, 77, 75, 73

6 The normal monthly amounts of rainfall for San Francisco and Denver are given below Find (1) the total amount of rainfall during a year in each city, (2) the arithmetic mean of the monthly rainfall for each, and (3) the standard deviation of the monthly rainfall for each

Denver 5, 6, 1 2, 1 9, 2 1, 1 4, 1 2, 1 3, 9, 1 0, 7, 5
San Francisco 4 0, 4 0, 2 8, 1 5, 6, 1, 0, 0, 1, 1 1, 2 3, 4 1

7 The heights of two basketball teams are given Compare the arithmetic means of the heights and also compare the standard deviations of the heights

Team A 6 ft , 6 ft , 6 ft 1 in , 5 ft 10 in , 6 ft 1 in
Team B 5 ft 7 in , 6 ft , 6 ft 5 in , 6 ft , 6 ft 3 in

Correlation [A]

Sometimes when we compare two measurable quantities we find a relationship existing between them such that a change in one tends to be associated with a change in the other The relationship may be definite enough to be called a functional relationship, or it may be less rigidly defined

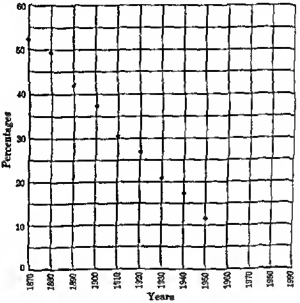
When we say, "The perimeter of a square is four times the length of one of its sides,' we are expressing a definite relationship—so definite a relationship that we can write the formula $P = 4s$ to express it Each time we assign a value to s in the formula, we get a definite value for P This relationship is functional

When we say, "As the years pass the percentage of workers engaged in agriculture becomes smaller," we are also expressing a relationship, but in this case the relationship is not exact enough to be represented by a formula For a specific year, say 1980, we cannot state with certainty the exact percentage of all workers who will be farmers The relationship in this case is not functional

To specify all such relationships, functional and otherwise, we use the word correlation A correlation expresses a tendency *toward* functionality

Curve Fitting [B]

Even though we cannot state all correlations with mathematical exactness, it is important, in many areas of living, to find approximate mathematical statements for those without exact ones In the graph below the percentages of all gainful workers, 10 years of age or older, engaged in agriculture have been indicated for each tenth year since 1870 Notice that the points of the graph form a very close approximation to a straight line—so close that we become interested in trying to write an equation as representative as possible of the points shown We call such a line a line of "best fit"

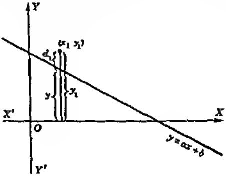


Percentage of Gainful Workers 10 Years of Age or Older in Agriculture 1870–1950

There are many ways of obtaining such a line We shall use the method of least squares

Let us suppose that (x_1, y_1), (x_2, y_2), , (x_n, y_n) represent points for which we are seeking a line of best fit, and let $y = ax + b$ represent that line While some of the points (x, y) may lie on the line, some points may not Let (x_1, y_1) represent a point which does not lie on

the line Then $y_1 - y$ represents the vertical *deviation* (d_1) of the point from the line, that is, $d_1 = y_1 - y$ Similarly, $d_2 = y_2 - y$, $d_3 = y_3 - y$, and so on represent the amounts of deviation of the other points from $y = ax + b$ In case a point lies on the line its deviation is zero



In the sense of least squares we say that $y = ax + b$ becomes the line of best fit when constants a and b are such that $d_1^2 + d_2^2 + \cdots + d_n^2$ is a minimum

Now, since any two points on the same vertical line have the same abscissa $x_1 = x$ Substituting x_1 for x in $y = ax + b$ gives $y = ax_1 + b$

$$\left.\begin{array}{ll} \text{Then} & d_1 = y_1 - (ax_1 + b) \\ \text{and} & d_2 = y_2 - (ax_2 + b) \\ & \cdots \\ & d_n = y_n - (ax_n + b) \end{array}\right\} \quad (1)$$

Squaring both members of each of the equations in (1) gives

$$\left.\begin{array}{l} d_1^2 = y_1^2 - 2\,ax_1y_1 - 2\,by_1 + a^2x_1^2 + 2\,abx_1 + b^2 \\ d_2^2 = y_2^2 - 2\,ax_2y_2 - 2\,by_2 + a^2x_2^2 + 2\,abx_2 + b^2 \\ \cdots \\ d_n^2 = y_n^2 - 2\,ax_ny_n - 2\,by_n + a^2x_n^2 + 2\,abx_n + b^2 \end{array}\right\} \quad (2)$$

Adding equations (2) gives

$$\Sigma d^2 = \Sigma y^2 - 2\,a\Sigma xy - 2\,b\Sigma y + a^2\Sigma x^2 + 2\,ab\Sigma x + nb^2 \quad (3)$$

If, for the moment, we consider b a constant, the right member of (3) becomes a quadratic function in terms of a, and we have

$$\Sigma d^2 = a^2\Sigma x^2 + 2\,a(b\Sigma x - \Sigma xy) + \text{other terms (a constant)} \quad (4)$$

As we have already seen on page 252, this quadratic function has a minimum value when

$$a = -\frac{2(b\Sigma x - \Sigma xy)}{2\,\Sigma x^2},$$

that is, when
$$a\Sigma x^2 + b\Sigma x = \Sigma xy \tag{5}$$

If, for the moment, we consider a as a constant, the right member of (3) becomes a quadratic function in b, and we have

$$\Sigma d^2 = nb^2 + 2\,b(a\Sigma x - \Sigma y) + \text{other terms (a constant)} \tag{6}$$

Now the right member of (6) reaches its minimum when

$$b = -\frac{2(a\Sigma x - \Sigma y)}{2\,n}$$

or
$$nb + a\Sigma x = \Sigma y \tag{7}$$

Hence, in order for $d_1^2 + d_2^2 + \quad + d_n^2$ to be a minimum, a and b must* satisfy equations (5) and (7), that is, the system

$$\begin{aligned} a\Sigma x^2 + b\Sigma x &= \Sigma xy \\ a\Sigma x + nb \quad &= \Sigma y \end{aligned} \tag{8}$$

Let us now use the procedure just outlined to find the equation of the best-fitting line for the information that produced the graph on page 481

Example The following table shows the percentage of all gainful workers, 10 years of age or older, engaged in agriculture in 1870 and in each tenth year since Find the equation which best represents the relationship between the years and the percentage of workers

Percentage of Gainful Workers 10 Years of Age and Over Employed in Agriculture 1870–1950

Year	1870	1880	1890	1900	1910	1920	1930	1940	1950
Percentage of Workers Employed in Agriculture	53.0	49.4	42.6	37.5	31.0	27.0	21.4	17.6	11.6

*While it can be proved that the condition stated is also sufficient the proof is beyond the scope of this book.

In the equation which we plan to write we shall let x represent the number of years and y the percentages. To ease the work of computation we shall let the year 1870 be represented by $x = 0$, the year 1880 by $x = 1$, and so on. We can now make a new table which contains all the information we shall need for substituting in the equations (8), page 483.

x	y	x^2	xy
0	53.0	0	0
1	49.4	1	49.4
2	42.6	4	85.2
3	37.5	9	112.5
4	31.0	16	124.0
5	27.0	25	135.0
6	21.4	36	128.4
7	17.6	49	123.2
8	11.6	64	92.8
$\Sigma x = 36$	$\Sigma y = 291.1$	$\Sigma x^2 = 204$	$\Sigma xy = 850.5$

From the totals in the table and the fact that $n = 9$, we substitute in the equations of (8). Thus

$$a\Sigma x^2 + b\Sigma x = \Sigma xy \quad \text{becomes} \quad a(204) + b(36) = 850.5$$
$$a\Sigma x + nb = \Sigma y \quad \text{becomes} \quad a(36) + 9b = 291.1$$

Simplifying, we have

$$\begin{aligned} 204a + 36b &= 850.5 \\ 36a + 9b &= 291.1 \end{aligned} \qquad (9)$$

Solving equations (9), we obtain $a = -5.23$, $b = 53.27$. When we substitute these values in $y = ax + b$, we have the line of best fit

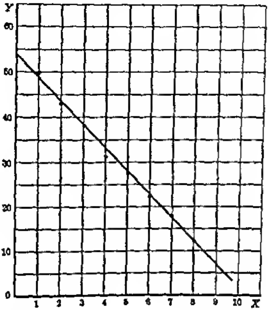
$$y = -5.23x + 53.27$$

In the graph on the opposite page this line of best fit is shown in color while the actual points located by the information given in the problem are shown in black.

We might call the line $y = -5.24x + 53.27$ a <u>trend line</u> because it shows the trend in the percentage of workers engaged in agriculture during the past 70 years.

Trend lines may be used to make predictions, but obviously such predictions must be made with caution. If the trend shown in the graph continues, it appears that farming will disappear by 1970—a situation which today seems impossible. Perhaps the line will soon

begin to slacken its downward course and move out to the right to form a curved line At present we cannot know, but the possibilities merit careful thinking on our part



Percentage of Gainful Workers in Agriculture 1870–1950

EXERCISES

[B]

1. Find the equation of the line of best fit for the following table of the weights of 18-year-old men of various heights

HEIGHT (inches)	63	64	65	66	67	68	69	70	71	72	73	74	75	76
WEIGHT (pounds)	124	127	131	136	141	146	150	154	158	163	167	171	174	177

HINT Use $x = 0$ for 63 inches, $x = 1$ for 64 inches, and so on

2 The following table shows the location in degrees and minutes west longitude of the center of population of the United States at the time of each census since 1790

DATE	LOCATION	
	Degrees	Minutes
1790	76	11
1800	76	56
1810	77	37
1820	78	33
1830	79	16
1840	80	18
1850	81	19
1860	82	48
1870	83	35

DATE	LOCATION	
	Degrees	Minutes
1880	84	39
1890	85	32
1900	85	48
1910	86	32
1920	86	43
1930	87	8
1940	87	22
1950	88	9

Notice that the westward trend continued at the rate of about 1 degree in 10 years from 1800 until about 1900, at which time it slowed to about 1 degree in 20 years Round off the given longitudes to the nearest degree and find the equation of the line of best fit for the years from 1790 to 1890 inclusive In a similar manner find the equation of the line of best fit for the years 1900 through 1950

By substituting $x = ?$ in the equation you have found for the years 1790–1890, estimate where the center of population would have been in 1950 had the 1790–1890 trend continued

Obviously, the 1900–1950 trend cannot continue indefinitely Explain Is it possible that the line might reach a stopping point? Is it possible that it might turn back east again?

3 The following table shows the speeds in miles per hour of the cars which have won the Indianapolis 500-Mile Automobile Race since 1946 Find the equation of the line of best fit for the information

Year	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955
mph	114.8	116.3	119.8	121.3	124.0	126.2	128.9	128.7	130.8	128.2

4 The following information shows the number of television sets produced in each year 1947–1954 Find the equation of the line of best fit

Year	Number of Sets
1947	250 000
1948	1 000 000
1949	3 000 000
1950	7,500 000

Year	Number of Sets
1951	5,600 000
1952	6,300 000
1953	7 300 000
1954	6,500 000

Measuring Correlation [8]

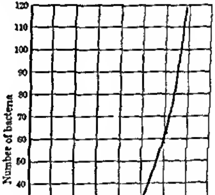
Do you recognize that a line of best fit provides a measure of correlation between the two quantities considered? For example, on page 484 the equation $y = -5.23x + 53.27$ shows that the percentage of workers in agriculture at any given time is approximately 53.27 more than -5.23 times the number of 10-year periods since 1870.

We shall content ourselves with the measures of correlation provided by the lines of best fit, though statisticians prefer a specific number known as the coefficient of correlation.

Non-Linear Correlation [8]

Obviously, the lines of best fit for some sets of numbers are curved rather than straight. If, for example, we plot the growth of cholera bacteria, we get the curve at the right. Finding the equation of the line of best fit in such cases is beyond the scope of this book.

Growth of Cholera Bacteria



Many sets of data which appear to produce lines of best fit which are straight, actually produce curved lines when longer periods of time are considered. For example, the downward trend in the number of agricultural workers shown in the graph on page 485 may, in future years, slacken and produce a curve.

Checking Your Understanding of Chapter 13

A chance to review

Before you leave this chapter make sure that you can

1 Make a frequency distribution (p 467) This includes being able to make a frequency distribution by classes (p 469)

2 Can find the median (p 468), the mode (p 468), and the arithmetic mean (p 468)

3 Can make a histogram (p 472) and a frequency polygon (p 472)

4 Can find the standard deviation (p 479)

5 Can spell and use correctly the following expressions

arithmetic mean (p 468)	median (p 468)
average (p 468)	mode (p 468)
correlation (p 480)	range (p 468)
distribution (p 468)	skew (p 476)
frequency polygon (p 472)	standard deviation (p 478)
histogram (p 472)	statistics (p 467)

If you expect to do better than average work, you will also want to make sure that you

6 Understand the meaning of normal probability (pp 473–476)

7 Can find a line of best fit when the data indicate a linear relationship (pp 481–486)

CHAPTER REVIEW

[A]

1 Make a frequency distribution for the following scores made by a fourth grade class on a spelling test

100, 95, 85, 90, 80, 85, 80, 95, 75, 85, 100, 80, 90, 90, 85, 70, 100, 80, 75, 70, 95, 85, 60, 90, 75, 80, 90, 85, 85, 80

2 State the range of the scores in Exercise 1

3 Find the median, the mode, and the arithmetic mean of the scores in Exercise 1

4 On five consecutive French tests Kay and Pat made scores as follows

Kay	100	90	85	80	70
Pat	85	80	85	80	85

a Find the arithmetic mean of the scores for each girl

b Find the standard deviation of the scores for each girl

5. To the nearest thousand the number of dwellings authorized between 1945 *and* 1948 *and* between 1950 and 1953 were

Year	Number of Dwellings	Year	Number of Dwellings
1945	161 000	1950	837 000
1946	528 000	1951	601 000
1947	508,000	1952	617 000
1948	531 000	1953	570 000

a Compare the arithmetic mean of the number of homes authorized between 1945 and 1948 with the arithmetic mean of the number authorized between 1950 and 1953

b Find the standard deviation of the numbers of authorizations for each four-year period

6. The highest temperatures registered in one town during each day for two months were

80, 81, 80, 83, 85, 84, 80, 78, 75, 74, 75, 73, 72, 72, 75, 80, 82, 80, 82, 80, 85, 86, 86, 85, 84, 86, 86, 88, 90, 92, 94, 94, 93, 90, 89, 89, 92, 93, 89, 89, 72, 72, 80, 80, 83, 84, 86, 86, 87, 88, 86, 85, 85, 80, 85, 85, 84, 83, 80, 83

a. Using classes of 5, make a frequency distribution of the temperatures

b Find the arithmetic mean of the temperatures according to the grouping used in part a

On your paper write the numbers 7 to 16 in column form After each number write T or F according to whether you think each of the following statements is true or false

7. A statistician's conclusions can be no more accurate than the information he uses to form the conclusions

8. A *distribution which does not* fit *into the bell shaped* normal distribution pattern is not a normal distribution

9 A median is a kind of average

10 The standard deviation for a set of data provides a measure of departure from the arithmetic mean of the data

11. A histogram is a vertical bar graph of a frequency distribution

12. A frequency polygon is a broken-line graph of a frequency distribution

13 The arithmetic mean of a set of data always serves as an accurate representative of the data

14 The arithmetic mean of a set of data is the middle item in the set

15. If John's scores on four successive tests were 80, 95, 100, and 100 while Bill's scores on the same tests were 50, 65, 70, and 50, John's scores showed the same range as Bill's

16 The statement "Anything can be proved by statistics" is true

[B]

Find the equations of the lines of best fit for the following sets of numbers

17

x	1	2	3	4	5
y	2	2	0	1	0

18

x	−1	1	2	3
y	−3	4	3	8

19.

x	−2	0	2	4	6
y	−1	0	1	2	3

20

x	−1	1	3	5
y	−5	3	12	20

21 In 5 tosses of a coin what is the normal probability that you will get zero heads, 1 head, 2 heads, 3 heads 4 heads, 5 heads?

CHAPTER TESTS

[Test A]

1 The ages listed for 50 boys entering college were

18, 18, 18, 19, 16, 19, 19, 17, 18, 18, 18, 19, 17, 20, 18, 18, 21, 19, 18, 23, 17, 19, 19, 17, 19 19, 18 19, 18, 18, 19, 18, 19, 19, 20, 19, 21, 18, 20, 18, 19, 19, 18, 18, 18, 19, 19, 18, 20, 18

a Make a frequency distribution of the ages

b State the range of the ages

c. Find the arithmetic mean of the ages

d Find the median of the ages

2. The scores made by the boys in Ex 1 on an entrance examination were

80, 90, 100, 70, 90, 110, 80, 100, 120, 80, 100, 100, 100, 60, 80, 90, 100, 110, 120, 90, 90, 90, 100, 90, 70, 80, 70, 110, 100, 120, 100, 110, 70, 90, 130, 100, 140, 90, 100, 120, 130, 90, 110, 90, 80, 100, 80, 120, 100, 90

a. Using groups of ten, make a frequency distribution of the scores

b Make a histogram of the scores

c Find the arithmetic mean of the scores using the frequency distribution

3 Find the standard deviation of the following highest temperatures recorded in an eastern city during one week

80, 75, 78, 82, 90

[Test B]

1 If you toss 6 coins, what is the probability that you will get each of the following numbers of heads 0, 1, 2, 3, 4, 5, and 6?

Find the equation of the line of best fit for each of the following sets of numbers

2

x	1	2	3	4	5
y	1	2	2	4	6

3

x	−2	2	3	5
y	2	2	4	4

THE BEGINNING OF THE CALCULUS

We attribute the invention of the calculus to Leibnitz and Newton, but some credit for the invention should be given to Fermat (1601–1665), the noted French mathematician, and to Archimedes who centuries before had used the underlying principles of the calculus. Many other mathematicians contributed to its development. Notable among these was Cauchy (1789–1857).

Sir Isaac Newton (1642–1727) is thought by many persons to be one of the three greatest mathematicians, the other two being Archimedes (287 B.C.–212 B.C.) and Gauss (1777–1855). As a child, Newton showed inventive ability by making mechanical toys. He prepared for college at Grantham Grammar School and received his B.A. degree from Cambridge in 1664.

As a foundation for his work he had the analytic geometry of Fermat and Descartes, the beginning of the calculus of Fermat, the fundamental laws of planetary motion of Kepler, and the first two of his three laws of motion from Galileo.

Newton

During the bubonic plague years 1664–1665 Newton had time for study and contemplation. In this short time he made remarkable progress, inventing the calculus, discovering the universal law of gravitation, and proving that light consists of all colors. Newton probably invented the differential calculus while studying the second of his three laws of motion (which can be expressed by the formula $F = ma$). He probably invented integral calculus while pondering over his problems.

Newton's ability was demonstrated one day when one of his friends gave him a problem which had baffled the mathematicians of Europe for six months. Although tired from a day's work, Newton sat down after his evening meal and not only solved this problem but another difficult one.

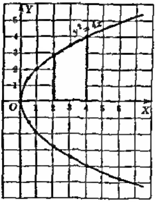
Gottfried Wilhelm von Leibnitz, the other inventor of the calculus, was born in Leipzig in 1646. He entered the University of Leipzig at the age

of 15 and was graduated at the age of 17. At first Leibnitz was interested in philosophy, and he was twenty-six years old before he concentrated on mathematics. It is remarkable that only five years later he published his work on the calculus. His work was similar to the calculus of Newton, which was not published. Leibnitz made the calculus usable by giving rules and formulas. He has been called the "Master of all trades" because he made noteworthy contributions to so many other fields, including logic and history.

Leibnitz

You can get a glimpse of the differential calculus by studying the next chapter. After you have completed the chapter, you can study the following example, which is solved by integral calculus.

Find the area included by the parabola $y^2 = 4x$, the x-axis, the line $x = 2$, and the line $x = 4$.



$$y^2 = 4x \quad \text{Then } y = 2x^{\frac{1}{2}}$$

$$A = \int_2^4 y\,dx = 2\int_2^4 x^{\frac{1}{2}}\,dx = 2\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_2^4$$

$$= \tfrac{4}{3}\left[x^{\frac{3}{2}}\right]_2^4 = \tfrac{4}{3}\left[4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = 6.89$$

We think of the area as being the sum of an infinite number of rectangles dx wide and y long. The derivative of $\frac{2}{3}x^{\frac{3}{2}}$ is $x^{\frac{1}{2}}$ and the integral of $x^{\frac{1}{2}}$ is $\frac{2}{3}x^{\frac{3}{2}}$. Integration is the inverse of differentiation.

CHAPTER 19

Rates of Change

In this chapter you will learn a little about a very important branch of mathematics, the calculus.

If you drive your automobile 200 miles in 5 hours, your average speed for the trip is 40 miles an hour, 3520 feet a minute, and $58\frac{2}{3}$ feet a second. The average speed is the average number of units of distance for one unit of time. In making the trip your speed was not uniform, probably varying from zero miles an hour to 50 miles an hour. Your speed at any instant, which is called the instantaneous speed, was approximately given by your speedometer.

If a man spends \$600 a year for rent, his average rate per month is \$50. If his rent is not changed during the year, his instantaneous rate is \$600 a year and \$50 a month.

If a boy's weight increased 18 pounds in 12 months, his average monthly increase was $1\frac{1}{2}$ pounds. His instantaneous increase in weight, no doubt, was changing all the time.

If an automobile manufacturer increased his daily output 120 cars in one year, the average monthly increase was 10 cars. The instantaneous rate of increase was not uniform and could not be determined at any time.

In each case above, the average rate of change was found by dividing the total change by the time of the change. Only when the change was uniform could the instantaneous rate of change be determined.

Let us consider the function $s = 16\,t^2$. When t changes from 1 to 3 (changes 2 units), s changes from 16 to 144 (changes 128 units). For the interval $t = 1$ to $t = 3$, the average rate of change of s with respect to t is $\frac{128}{2}$, or 64. For this interval the instantaneous rate of change of s with respect to t varies from 32 to 96, as you will be able to verify later. The instantaneous rate of change of a function can be determined by a branch of mathematics known as *calculus*.

Review of Functions[A]

If two variables are so related that when a value is assigned to the independent variable a corresponding definite value (or set of values) of the other is determined, the dependent variable is a *function* of the independent variable.

From this definition it follows that any change in the independent variable usually causes a change in the dependent variable.

In the equation $y = 3\,x^2 - x + 6$, x is the independent variable and both y and the expression $3\,x^2 - x + 6$ are functions of x. The equation $y = 3\,x^2 - x + 6$ can be written $f(x) = 3\,x^2 - x + 6$ and $y = f(x) = 3\,x^2 - x + 6$.

Example 1 $f(x) = 3x^2 - x + 6$ Find $f(3)$

Solution $f(3) = 3(3)^2 - 3 + 6 = 30$

Example 2 $f(x) = 3x^2 - x + 6$ Find $f(x + 1)$

Solution $f(x + 1)$ means that $x + 1$ is to be substituted for x in the function

Then $$f(x+1) = 3(x+1)^2 - (x+1) + 6$$
$$= 3x^2 + 5x + 8$$

EXERCISES [A]

1 $f(x) = 3x + 4$ Find $f(2)$, $f(0)$, $f(-3)$

2 $f(x) = x^2 + 4x - 8$ Find $f(5)$, $f(-2)$, $f(0)$

3 $f(x) = 2x^2 + 5x$ Find $f(h)$, $f(-h)$

4 $f(t) = 8t^2 - 3t$ Find $f(a)$, $f(-b)$, $f(c^2)$

5 $f(x) = 3x - 6$ Find $f(\Delta x)$

NOTE Δx means a small increase in x It is read "delta x"

6 $f(x) = 10 - 4x$ Find $f(\Delta x)$

7 $f(x) = x^2 - x$ Find $f(x - 3)$

8 $f(x) = 2x^2 + x$ Find $f(x + \Delta x)$

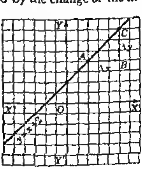
9 $f(x) = x^2 - 6x$ Find $f(x + \Delta x)$

10 $f(x) = x^2 - 2x$ Find
(a) $f(x + \Delta x)$, (b) $f(\Delta x)$, (c) $f(x + \Delta x) - f(x)$

11 $f(x) = x^3$ Find $f(x + \Delta x) - f(x)$

Average Rates of Change of Functions [A]

The average rate of change of a function for any given interval is equal to the change of the function divided by the change of the independent variable in the interval



Example 1 Find the average rate of change of $y = x + 2$ in the interval $x = 2$ to $x = 5$

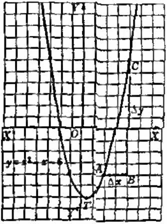
Solution If $x = 2$ $y = 4$
If $x = 5$, $y = 7$

The change in $x = \Delta x = 5 - 2 = 3$ and the change in $y = \Delta y = 7 - 4 = 3$ In the diagram, AB is the change in x and BC is the

change in y. The average rate of change of the function in the interval $x = 2$ to $x = 5$ is $\frac{3}{3}$ or 1. You learned in Chapter 5 that the rate of increase of a linear function is constant, so the average rate of change of the function in any other interval is 1. Since the rate of increase of the function is constant, the instantaneous rate of change of the function is 1.

The slope of the line AC is 1, since the slope of the line is equal to the function increase divided by the x increase.

Example 2 Find the average rate of change of the function $y = x^2 - x - 6$ in the interval $x = 2$ to $x = 4$.



Solution If $x = 2$, $y = -4$.
If $x = 4$, $y = 6$.

We see that the change in $y = \Delta y = 6 - (-4) = 10$, and the change in $x = \Delta x = 4 - 2 = 2$. Then the average rate of change in the interval $x = 2$ to $x = 4$ is $\frac{10}{2}$ or 5. This means that for the given interval the function changes 5 times as much as x. In the diagram AB is the change in x, and BC is the change in the function.

It can be seen from the graph that the average rate of change of the function is not the same for all intervals. For example, the average rate of change of the function in the interval $x = 1$ to $x = 2$ is 2 because $\frac{\Delta y}{\Delta x} = \frac{-4 - (-6)}{2 - 1} = 2$.

(A) **EXERCISES**

1 From the graph of $y = x + 2$ on page 496 find:

a The value of y when $x = -4$, $x = 0$, $x = 3$

b The average rate of change of the function in the interval $x = -2$ to $x = 4$

2 From the graph of $y = x^2 - x - 6$ above find:

a The average rate of change of y in the interval $x = 2$ to $x = 3$

b The average rate of change of y in the interval $x = -2$ to $x = 0$. What does the sign of this rate of change indicate?

3. $f(x) = 2x^2 - 6x + 3$. Find the average rate of change of the function for the interval $x = 1$ to $x = 2$.

4. $y = 4x - x^2$. Find the average rate of change of the function and the slope of the secant for the interval $x = 3$ to $x = 3.1$. (A secant is a line that intersects a curve in two or more points.)

5. $f(x) = 3x^2 - 2$. Find the slope of the chord that joins the points of the graph whose abscissas are -1 and -2.

6. $s = 16t^2$. Find the average rate of change of s in the interval $t = 3$ to $t = 5$.

7. $y = 8 - 2x^2$. Find the slope of the secant that intersects the graph where $x = 0$ and $x = 2$.

8. Draw the graph of the function $\frac{1}{4}x^2 - 2$. Draw the secant through the points of the curve whose abscissas are 4 and 6. Draw the lines that show the values of the function when $x = 4$ and $x = 6$. What is the slope of the secant? What is the average rate of change of the function in the interval from $x = 4$ to $x = 6$?

9. Draw the graph of $y = \frac{1}{2}x^2$. Make and complete a table like the following.

FIXED POINT		MOVING POINT		CHANGE IN x	CHANGE IN y	SLOPE OF SECANT
x	y	x	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
1	0.5	1.1	0.605	0.1	0.105	1.05
1	0.5	1.01	?	?	?	?
1	0.5	1.001	?	?	?	?

As the interval gets smaller and smaller, to what number does the slope of the secant become nearer and nearer? As the moving point of the curve approaches the fixed point as a limit, what value does the slope of the secant approach as a limit? Estimate the instantaneous rate of change of the function when $x = 1$.

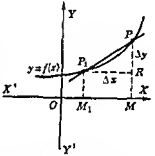
10. $x^2 + y^2 = 25$ is the equation of a circle whose center is at the origin and whose radius is 5. Find the average rate of

change of the function in the interval $x = 3$ to $x = 3.1$ What is the slope of the chord for this interval?

Formula for the Average Rate of Change of a Function[A]

The average rate of change of a function equals the increase in the function divided by the increase in the independent variable

At the right is part of the graph of $y = f(x)$ Let P_1 and P denote two points on the curve The interval is from $x = OM_1$ to $x = OM$ As x increases from OM_1 to OM, y increases from MR to MP Then P_1R is the x increase and RP is the y increase Let Δy represent the y increase and Δx represent the x increase Then for the interval Δx the average rate of change of the function is $\frac{\Delta y}{\Delta x}$ The slope of the secant P_1P is also $\frac{\Delta y}{\Delta x}$



Let (x_1, y_1) be the co-ordinates of P_1 Then the abscissa of the point $P = OM_1 + M_1M = x_1 + \Delta x$ The value of y at $P_1 = f(x_1)$ and the value of y at $P = f(x_1 + \Delta x)$ Then $\Delta y = f(x_1 + \Delta x) - f(x_1)$

Then $$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

This formula gives the average rate of change of the function and the slope of the secant P_1P for the interval Δx You should remember this formula and be able to develop it at any time because of its importance in differential calculus

Example 1. Find the average rate of change of the function $y = 3x^2 - 5x$ in the interval $x = 1$ to $x = 1.1$

Solution.
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$
$$x_1 = 1,\ x_1 + \Delta x = 1.1, \text{ and } \Delta x = 0.1$$
$$f(x) = 3x^2 - 5x$$
$$f(x_1 + \Delta x) = f(1.1) = 3(1.1)^2 - 5(1.1) = -1.87$$
$$f(x_1) = f(1) = 3(1)^2 - 5(1) = -2$$
$$\Delta y = -1.87 - (-2) = 0.13$$
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{0.13}{0.1} = 1.3$$

EXERCISES

[A]

Use the formula to find the average rate of change of each of the following functions in the interval given

1 $y = 2x^2 + 3$ in the interval $x = 4$ to $x = 5$

2 $y = 3x^2 - x$ in the interval $x = 0$ to $x = 1$

3 $y = x^2 - 7x + 1$ in the interval $x = 3$ to $x = 3.1$ What is the slope of the secant passing through the points of the graph of this function, whose abscissas are 3 and 3.1?

4 $y = 10 - 3x^2$ in the interval $x = 2$ to $x = 2.2$ What is the slope of the secant of the graph of $y = 10 - 3x^2$, which passes through the points whose abscissas are 2 and 2.2?

Example 2 Find the average rate of change of the function $y = 3x^2 - 2$ for the increment Δx starting at the point (x_1, y_1)

Solution Δx means a small increase in x

$$f(x) = 3x^2 - 2$$
$$f(x_1 + \Delta x) = 3(x_1 + \Delta x)^2 - 2$$
$$f(x_1) = 3x_1^2 - 2$$
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{3(x_1 + \Delta x)^2 - 2 - 3(x_1)^2 - (-2)}{\Delta x}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{3x_1^2 + 6x_1\,\Delta x + 3(\Delta x)^2 - 3x_1^2}{\Delta x}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = 6x_1 + 3\,\Delta x$$

The rate of change of the function when x has any value x_1 is equal to 6 times the value x_1 increased by 3 times a small value of x For $x_1 = 3$ and $\Delta x = 1$, $\frac{\Delta y}{\Delta x} = 6(3) + 3(1) = 21$ Also, the slope of the secant through the points where $x = 3$ and $x = 4$ is 21

EXERCISES

[A]

1 $f(x) = x^2 - 12$ Find $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

2 $y = 8 - 6x^2$ Find $\frac{\Delta y}{\Delta x}$

3 $y = 2x^2 - x - 9$ Find $\frac{\Delta y}{\Delta x}$ Find the average rate of change of the function in the interval $x = -2$ to $x = 0$

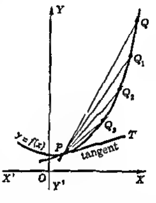
4. $s = 16\,t^2$. Find $\frac{\Delta s}{\Delta t}$.

5. $s = t^2 - 4\,t$ Find the average rate of change of s with respect to t in the interval $t = 4$ to $t = 4.1$

6 Find the slope of the secant of $y = 4\,x^2 - x + 1$ through the points (0, 1) and (0.1, 0.94)

Tangent to a Curve [A]

A secant of a curve is a line through two or more points of the curve In the accompanying figure PQ is a secant of the graph of $y = f(x)$ Suppose that the point P is fixed and that the point Q is free to move along the curve Let it move along the curve toward the point P As it does so, the secant changes direction At one time the secant takes the position PQ_1, at another time it takes the position PQ_2, and at another time it takes the position PQ_3 As Q continues to move along the curve toward P, the secant takes a direction near that of PT, the tangent to the curve at P As Q approaches P as a limit, the secant PQ approaches the tangent PT as a limit



A tangent to a curve is the limiting position of a secant when one of its points of intersection with the curve approaches the other as a limit How does this definition of a tangent differ from the definition used in plane geometry?

Derivatives [A]

The average rate of change of the function $y = f(x)$ for the interval Δx is given by the formula

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

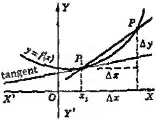
We are especially interested in the instantaneous rate of change of the function $f(x)$ For the instantaneous rate of change of the function, both Δx and Δy are zero Then the quotient $\frac{\Delta y}{\Delta x}$ takes the indeterminate form $\frac{0}{0}$, which has no definite value

If the fraction $\frac{\Delta y}{\Delta x}$ has a limiting value when Δx approaches zero as its limit, this limiting value of $\frac{\Delta y}{\Delta x}$ denotes the instantaneous rate of change of the function when $x = x_1$.

The limiting value of $\frac{\Delta y}{\Delta x}$ as $\Delta x \longrightarrow 0$ is called the derivative of y with respect to x for $x = x_1$. The derivative of y with respect to x is denoted by the symbols $\frac{dy}{dx}$, $D_x y$, and $f'(x)$. Since x_1 can be any value of x, we can define the derivative of y with respect to x as follows

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let us now study the graph of $y = f(x)$. Let P_1P be a secant passing through the fixed point $P_1(x_1, y_1)$ and a movable point $P(x, y)$ of the curve $y = f(x)$. $\frac{\Delta y}{\Delta x}$ represents the slope of P_1P. As P moves along the curve toward P_1 and approaches P_1 as a limit, both Δx and Δy approach zero as a limit. When P coincides with P_1, the secant becomes a tangent, both Δx and Δy become zero, and the slope of P_1P becomes $\frac{0}{0}$, which is indeterminate.



For the continuous curve $f(x)$, $\frac{\Delta y}{\Delta x}$ has a limiting value as Δx approaches zero as a limit. The limiting value of $\frac{\Delta y}{\Delta x}$ as $\Delta x \longrightarrow 0$ represents the slope of the tangent at P_1.

From the above discussion it follows that

1. The derivative of the function $f(x)$ for the value $x = x_1$ gives the instantaneous rate of change of the function when $x = x_1$.
2. The derivative of the function $f(x)$ for the value $x = x_1$ gives the slope of the tangent to the curve $y = f(x)$ at the point (x_1, y_1).

Example Find the rate of change of the function $y = x^2 - 4x + 4$ and the slope of the tangent to its graph at the point (x_1, y_1)

Solution

$$\begin{aligned}
y &= x^2 - 4x + 4 \\
f(x + \Delta x) &= (x + \Delta x)^2 - 4(x + \Delta x) + 4 \\
\Delta y &= f(x + \Delta x) - f(x) \\
&= [(x + \Delta x)^2 - 4(x + \Delta x) + 4] - [x^2 - 4x + 4] \\
\Delta y &= x^2 + 2x\,\Delta x + (\Delta x)^2 - 4x - 4\,\Delta x + 4 - x^2 + 4x - 4 \\
\Delta y &= 2x\,\Delta x + (\Delta x)^2 - 4\,\Delta x \\
\frac{\Delta y}{\Delta x} &= 2x + \Delta x - 4
\end{aligned}$$

We now find the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero as a limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4)$$

$$\frac{dy}{dx} = 2x - 4$$

The derivative $2x - 4$ tells us that

1 For any value of x the instantaneous rate of change of the function with respect to x is 4 less than twice this value of x For example, when $x = 5$, the function changes 6 times as fast as x

2 The slope of the tangent at any point of the graph of the function is 4 less than twice the value of x, the abscissa of the point

The derivative of any other function expresses like facts

To Find the Derivative of $y = f(x)$

1. Find the value of the function when x has been given an increment That is, find $f(x + \Delta x)$
2 Find the increment of the function That is, find $f(x + \Delta x) - f(x)$
3 Find the average rate of change in the interval That is, find

$$\frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

4 Obtain the derivative by finding the limit of the quotient in step 3 as Δx approaches zero as a limit

[A]

EXERCISES

Find the derivatives of the following

1 $y = 3x - 6$
2 $y = x^2 + 4$
3 $y = 3x^2 - 4x + 1$
4 $f(x) = 2x^3 + 3x$
5 $y = x^2 - x + 1$
6 $s = 10t^2$
7 $y = \frac{1}{3}x^3 + 4$
8 $2y = 4x^2 - 10x$

9 Find the slope of the tangent to the curve whose equation is $y = 6x^2 - 4$ at the point $(1, 2)$

10 Find the slope of the tangent to the parabola $y = 4x^2 - 1$ at the point $(2, 15)$

Differentiation [A]

The process of finding the derivative of a function is called differentiation. We shall now prove, or accept as true without proof, some facts which will shorten and simplify the differentiation of algebraic functions.

The Derivative of x^n [A]

Let $y = f(x) = x^n$, n being a positive integer.

Since

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ and } f(x) = x^n$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

Expanding $(x + \Delta x)^n$ by the binomial theorem, we have

$$\frac{\Delta y}{\Delta x} = \frac{\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}(\Delta x)^2}{2!} + \cdots + (\Delta x)^n\right] - x^n}{\Delta x}$$

Then

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)x^{n-2}(\Delta x)}{2!} + \cdots + (\Delta x)^{n-1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[nx^{n-1} + \frac{n(n-1)x^{n-2}(\Delta x)}{2!} + \cdots + (\Delta x)^{n-1}\right]$$

Except for the first, all terms of the expression enclosed by brackets contain the factor Δx. Then as $\Delta x \longrightarrow 0$, all terms of the expression, except the first, vanish.

Then $$\frac{dy}{dx} = nx^{n-1}, \quad \text{or} \quad \frac{d(x^n)}{dx} = nx^{n-1}$$

In the preceding proof n is taken as a positive integer. We shall accept the conclusion as true for all real values of n.

The formula can be expressed by the following rule:

> **The derivative of a power of a variable**
> **with respect to the variable**
> **equals the power times**
> **the variable with an exponent 1 less than the power.**

Example 1. $y = x^5$. Find $\frac{dy}{dx}$.

Solution.

$$y = x^5$$
$$\frac{dy}{dx} = 5x^4$$

Example 2. $y = \frac{1}{x^4}$. Find $D_x y$.

Solution.

$$y = x^{-4}$$
$$D_x y = -4x^{-5}$$
$$= -\frac{4}{x^5}$$

Example 3. $f(x) = x$. Find $f'(x)$.

Solution.

$$f(x) = x$$
$$f'(x) = 1x^0 = 1$$

[A] **EXERCISES**

Find the derivatives of the following:

1. $y = x^4$
2. $y = x^3$
3. $y = x^7$
4. $y = x^9$
5. $y = x^{10}$
6. $y = \frac{1}{x^2}$
7. $s = t^3$
8. $m = \frac{1}{n^2}$
9. $y = \frac{1}{x^5}$

The Derivative of a Constant[A]

Let $y = c$. Then $y = cx^0$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$\frac{\Delta y}{\Delta x} = \frac{c(x + \Delta x)^0 - cx^0}{\Delta x}$$

Since $x^0 = 1$ and $(x + \Delta x)^0 = 1$,

$$\frac{\Delta y}{\Delta x} = \frac{c - c}{\Delta x} = 0$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} 0 = 0$$

Therefore,

> The derivative of a constant is zero

The Derivative of a Sum[A]

We shall accept as true without proof the following rule

> The derivative of an algebraic sum of a finite number of terms is the algebraic sum of their derivatives

Example 1 $y = x^3 + x^2 - x$

Solution $\frac{dy}{dx} = 3x^2 + 2x - 1$

Example 2 $f(x) = x^4 - x^2 + \frac{1}{x}$ Find $f'(x)$

Solution

$$f(x) = x^4 - x^2 + x^{-1}$$

$$f'(x) = 4x^3 - 2x - x^{-2}$$

$$= 4x^3 - 2x - \frac{1}{x^2}$$

EXERCISES

Find the derivatives with respect to x of the following

1 $y = x$	5 $y = x^2$	9 $y = x^3 - x$
2 $y = 1 - x$	6 $y = -x^3$	10 $y = x^2 - x$
3 $y = x + 3$	7 $y = -x + 1$	11 $y = \frac{1}{x} - \frac{1}{x^2}$
4 $y = -x$	8 $y = x^2 - x$	12 $y = x^5 - x$

The Derivative of a Product[A]

We shall accept as true the following

> The derivative of a product of a finite number of factors
> is the sum of the products obtained
> by multiplying the derivative of each factor
> by the product of the remaining factors

Example 1 $y = 4x^3$ Find $D_x y$

Solution. $D_x y = 4 \times (\text{derivative of } x^3) + x^3 \times (\text{derivative of } 4)$

$$= 4(3x^2) + x^3(0)$$
$$= 12x^2$$

Do you see that the derivative of "a constant times x^n" equals the constant times the derivative of x^n?

Example 2 $y = (x-3)(x-4)$ Find $\frac{dy}{dx}$

Solution 1
$$y = (x-3)(x-4)$$
$$y = x^2 - 7x + 12$$
$$\frac{dy}{dx} = 2x - 7$$

Solution 2
$$y = (x-3)(x-4)$$
$$\frac{dy}{dx} = (x-3)\frac{d(x-4)}{dx} + (x-4)\frac{d(x-3)}{dx}$$
$$= (x-3)(1) + (x-4)(1)$$
$$= 2x - 7$$

(A) **EXERCISES**

Find the derivatives

1. $y = 3x + 6$
2. $y = 4x^2 - 7x$
3. $y = 5x^3 - 7x + 1$
4. $y = x(x-4)$
5. $y = 7x^2 - 8x$
6. $y = x^2(x-1)$
7. $y = 4x^8 - 7x^7 + 3$
8. $y = \frac{1}{2}x - \frac{5}{x}$
9. $y = ax^2 - bx + c$
10. $s = t^2 + 4t$
11. $A = \pi r^2$
12. $A = p(1 + .04)$

The Derivative of a Fraction (A)

We shall accept the following as true without proof

The derivative of the fraction $\frac{u}{v} = \frac{vD_xu - uD_xv}{v^2}$.

Example 1 $y = \frac{1}{x^3}$. Find $\frac{dy}{dx}$.

Solution 1
$$\frac{dy}{dx} = \frac{x^3 D_x 1 - 1\, D_x x^3}{x^6}$$
$$= \frac{0 - 3x^2}{x^6} = \frac{-3}{x^4}$$

Solution 2

$$y=\frac{1}{x^3}$$
$$y=x^{-3}$$
$$\frac{dy}{dx}=-3x^{-4}=\frac{-3}{x^4}$$

Example 2 Find the derivative of $\dfrac{2x-1}{x^2+2}$

Solution

$$y=\frac{2x-1}{x^2+2}$$
$$\frac{dy}{dx}=\frac{(x^2+2)(2)-(2x-1)(2x)}{(x^2+2)^2}$$
$$\frac{dy}{dx}=\frac{-2x^2+2x+4}{(x^2+2)^2}$$

The Derivative of a Function of a Function [A]

If y is a function of u and u is a function of x,
the derivative of y with respect to x
is the product of the derivative of y with respect to u
and the derivative of u with respect to x

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

We shall not prove the above fact The theorem merely states that if y changes m times as fast as u, and u changes n times as fast as x, then y changes mn times as fast as x

Example $y=\sqrt{4x-x^2}$ Find D_xy

Solution Let $u=4x-x^2$

Then $y=\sqrt{u}=u^{\frac{1}{2}}$

$$\frac{du}{dx}=4-2x$$
$$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}=\frac{1}{2\sqrt{4x-x^2}}$$
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{4x-x^2}}\cdot(4-2x)=\frac{4-2x}{2\sqrt{4x-x^2}}=\frac{2-x}{\sqrt{4x-x^2}}$$

[A]

EXERCISES

Find the derivatives with respect to x

1 $y=\dfrac{1}{x}$

2 $y=\dfrac{3}{x}$

3 $y=\dfrac{2x-3}{x^2-1}$

4 $y=\dfrac{1-6x^2}{x^3}$

5 $y=\dfrac{x+x^2}{x^3-1}$

6 $y=\dfrac{x-1}{x^3+1}$

Find the derivatives of the following functions

7 $y = x^8$

8 $y = 5x^3$

9 $y = 7x^{10}$

10 $f(x) = \frac{4}{x}$

11 $f(x) = 1 - 7x^3$

12 $y = -\frac{1}{x^4}$

13. $f(x) = x(x+5)$

14. $s = 4t^2 - t$

15 $2y = 6x^3$

16 $y = \frac{1}{6}(x^2 - x)$

17 $y = \sqrt{x^2 - 4}$

18 $y = \sqrt{2x^3 - 5}$

19. $y = \frac{x}{x-7}$

20 $y = \frac{3x-1}{2x+4}$

21 $y = \frac{x-3}{1-5x}$

22 $y = (x-3)(3+x)$

23 $y = 1 - \frac{1}{x^2}$

24 $y = 5\sqrt{x}$

25 $y = \sqrt[3]{x^2 + x}$

26 $y = x\sqrt{x}$

27 $y = \sqrt{x}\sqrt[3]{x}$

28 $y = (x^2 - 5)(x + 2)$

29 $y = 7(x-1)^2$

30 $s = 30t - 8t^2$

31 $A = 2x\sqrt{60 - x^2}$

The Slope of a Curve[A]

The direction of a curve at any point on it is the direction of the tangent to the curve at that point The truth of this statement may be verified by observing sparks as they fly off the surface of a high-speed emery wheel Since the derivative of a function gives the slope of the tangent to the graph of the function, it also gives the slope of the curve at the point of tangency

[A] **EXERCISES**

Find the slopes of the following curves at the points indicated

1 $y = 3x^2 - x$ at $(3, 24)$

2 $y = x^2 - 6x + 3$ at $(2, -5)$

3. $y = 4x^2 - 5x$ at $(-1, 9)$

4. $y = 10 - x^2$ at $(0, 10)$

5 $y = x - 5x^2$ at $(\frac{1}{10}, 0.05)$

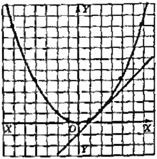
6. $s = 10t - t^2$ at $(1, 9)$

7. $y = x^2 - 6x + 3$ Find the value of x for which the slope of the curve is 2, -3, 0

8. Find the slope of the curve $y = x^3 - x^2 + x$ at the point where $x = 2$, where $x = -\frac{4}{3}$

The Equation of the Tangent to a Curve[A]

The point-slope form (p 149) of an equation can be used to find the equation of the tangent to any continuous curve at any point on the curve To do this we find the slope m of the curve at this point and substitute the values of x_1, y_1, and m in the equation $y - y_1 = m(x - x_1)$



Example Form the equation of the tangent to the curve $y = \frac{1}{4}x^2$ where $x = 2$

Solution $y = \frac{1}{4}x^2$

$$\frac{dy}{dx} = \frac{1}{4}(2x) = \frac{1}{2}x$$

For $x = 2, y = 1$, and $\frac{dy}{dx} = 1$

$$y - y_1 = m(x - x_1)$$

Then $y - 1 = 1(x - 2)$

Simplifying, $x - y = 1$, the equation of the tangent at $(2, 1)$

EXERCISES [A]

Form the equations of the tangents to the following curves at the points indicated

1 $y = 2x^2 - x + 3$ at $(1, 4)$

2 $y = x^2 - 4x$ at $(0, 0)$

3 $y = 10 - x^2$ at $(-2, 6)$

4. $y = 4 - 3x^2$ at $(1, 1)$

5 $y = \frac{1}{2}x^2$ at $(2, 2)$

6 $y = (x - 3)(x + 1)$ at $(3, 0)$

7 $y = 6 - 3x^2$ at $(1, 3)$

8. $y = 5x^2 + x$ at $(-1, 4)$

9 $y = \frac{1}{2}x^2 + x$ at $(0, 0)$

10 $y = 3x^3 - 2x$ at $(1, 1)$

11 $xy = 4$ at $(2, 2)$

12. $xy = 36$ at $(6, 6)$

13 Find the equations of the tangents to the curve $x^2 + y^2 = 25$ at the points where the curve intersects the y-axis

14. Find the equation of the tangent to the curve $y^2 = 100 - x^2$ at the point $(6, 8)$

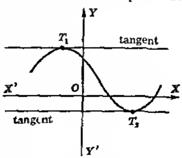
15. $y = x^2 - 4x$ Find the equations of the tangents to the curve at the points where the curve intersects the x-axis

Maximum and Minimum Values of Functions[A]

In Chapter 9 you learned how to find the maximum and minimum values of quadratic functions You learned that a curve is concave

downward at a maximum value of the function and concave upward at a minimum value of the function

In the figure, T_1 and T_2 are turning points of the curve (This curve is not the graph of a quadratic function.) At T_1 the function has a maximum value and at T_2 it has a minimum value Since the slope of a tangent at a turning point is zero, the value of the derivative at a turning point is zero



The derivative at T_1 changes from positive to negative, and at T_2 it changes from negative to positive

To Find the Maximum or Minimum Value of a Function

1. Find the derivative of the function.
2. Set the derivative equal to zero
3. The root (or roots) of the resulting equation is the abscissa of a turning point of the curve if the derivative changes sign at the point
4. Plot other points to the left and right of the point (or points) to determine whether the curve is concave upward or concave downward at the turning point
5. If the curve is concave downward at the turning point, the function has a maximum value, if the curve is concave upward at the turning point, the function has a minimum value

Example 1 Find the turning point of $y = 3x^2 - 6x + 4$ and determine whether the point is concave upward or downward

Solution $y = 3x^2 - 6x + 4$

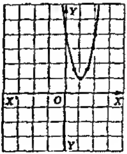
$$\frac{dy}{dx} = 6x - 6$$

Let $6x - 6 = 0$

Solving $x = 1$

For $x = 1, y = 1$, for $x = 0, \frac{dy}{dx}$ is negative, and for $x = 2, \frac{dy}{dx}$ is positive

Then the curve has a turning point at (1, 1) By plotting the points



$(1\frac{1}{2}, 1\frac{3}{4})$ and $(\frac{1}{2}, 1\frac{3}{4})$ we find that the curve is concave upward. What other fact tells us that the curve is concave upward at $(1, 1)$? How did you determine the kind of concavity in Chapter 9?

Example 2 Find the maximum value of $y = x^3 - 12x + 8$

Solution $\quad y = x^3 - 12x + 8$

$$\frac{dy}{dx} = 3x^2 - 12$$

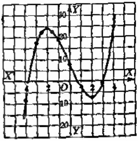
Let $3x^2 - 12 = 0$

Solving $\quad x = \pm 2$

If $x = 2$, $y = -8$; if $x = -2$, $y = 24$

By plotting points on both sides of the points $(2, -8)$ and $(-2, 24)$ we find that the function has a maximum value at $(-2, 24)$ and a minimum value at $(2, -8)$

The maximum value of $x^3 - 12x + 8 = 24$



(A)

EXERCISES

Find the turning points of

1 $y = x^2$

2 $y = x^2 - 10x$

3 $y = 2x^3 + 3x^2 - 4$

4 $y = 3 - x - x^2$

5 $f(x) = 4x^2 - 20x$

6 $f(x) = 4x^3 - 5x^2$

7 Show that the turning point of $y = ax^2 + bx + c$ is at $x = -\frac{b}{2a}$. Can you remember this fact?

Find the minimum value of

8 $y = x^2 + x - 5$

9 $y = 2x^2 - x + 1$

10 $y = x^2 - 16x + 3$

11 $y = x^2 - 8x$

12 $y = 40x^2$

13 $y = \frac{1}{2}x^2 - 11x$

Find the maximum value of each of the following

14 $y = 4 - x^2$

15 $y = 6x - x^2$

16 $y = -x^2 - 8$

17 $y = -4x^2 + 2$

18 $y = 4 - 10x^2$

19 $y = x^3 - 12x + 4$

Find the maximum or minimum value of each of the following

20 $y = 2x^2 - x$

21 $y = 6x - \frac{1}{2}x^2$

22 $y = 5x^2 - 30x$

23 $y = \frac{1}{2}x^2 + 5x$

Find the turning points of the graphs of the following, and state the maximum and minimum values

24. $y = x^3 - x^2$ 26. $y = x^3 - 27x + 18$

25. $y = 2x^3 - 15x$ 27. $y = x^3 - \frac{1}{2}x^2 - 10x$

Problems Involving Maxima and Minima [A]

The derivative is a valuable tool in solving problems in which a maximum or a minimum value is sought

Example An open rectangular box is to be made from sheet metal 8 inches wide and 12 inches long by cutting out squares at the corners and folding, as along the dotted lines shown in the diagram Find the size of the squares when the box has the largest volume

Solution Let V = the volume of the box
and let x = a side of each of the squares

Then
$$V = x(12 - 2x)(8 - 2x)$$
$$V = 4x^3 - 40x^2 + 96x$$
$$D_xV = 12x^2 - 80x + 96$$

For V to be a maximum or minimum, we set
$$12x^2 - 80x + 96 = 0$$

Solving
$$x = \frac{10 \pm 2\sqrt{7}}{3}$$
$$= 5.097 \quad \text{or} \quad 1.569$$

The first answer is impossible The maximum capacity of the box is obtained when the squares are 1.569 inches on a side

[A] **PROBLEMS**

1 Separate 32 into two parts such that their product is a maximum $p = x(32 - x)$

2 Separate 40 into two parts such that their product is the largest possible

3 Separate 20 into two parts such that the sum of their squares is a minimum

4 Separate 10 into two parts such that the sum of their cubes is a minimum

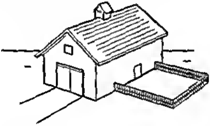
5 Show that the largest rectangle having a perimeter of 40 inches has an area of 100 square inches

6 Show that the area of the largest rectangle having a perimeter of 16 inches is 16 square inches

7 The legs of an isosceles triangle are 12 inches each Find the length of the base when the triangle has the maximum area $(A = x\sqrt{144 - x^2})$

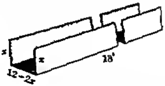
8 Two sides of a triangle are 8 inches each What is the length of the third side of the triangle when the triangle has the maximum area?

9 Some pupils in a sheet metal shop wished to cut corners of a square piece of sheet metal 20 inches on a side, fold the edges, and make an open rectangular box of maximum capacity How large should each of the four squares be? $[V = x(20 - 2x)(20 - 2x)]$



10 A man has 60 feet of wire fencing with which to enclose 3 sides of a chicken yard, the fourth side being a side of a barn What is the size of the largest yard he can have?

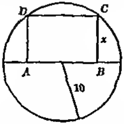
11 The hypotenuse of a right triangle is 10 inches Find the legs of the triangle that has the maximum area



12 A sheet metal worker wishes to shape a sheet of metal into a trough as shown The sheet metal is 12 inches wide and 18 feet long How much metal should he turn up at each side so that the trough may have the maximum capacity?

13 Find the positive number which when it is added to its reciprocal will give the smallest sum

14. When the radius of a circle is changing at the rate of 4 inches a second, what is the rate of change in the area when the radius is 6 inches?



15 $ABCD$ is a rectangle inscribed in a semicircle with radius 10 inches Find the maximum area of the rectangle

(B)

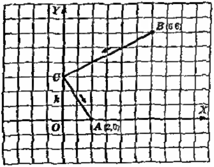
16 The volume of a cylindrical tin can is expressed by $V = \pi r^2 h$ and the surface of the can is expressed by $A = 2\,\pi r(r + h)$ Find the dimensions of a quart tin can which has the least amount of metal

SUGGESTION 1 gal = 231 cu in Solve $\pi r^2 h = \frac{231}{4}$ for h, and substitute this value of h in the formula $A = 2\,\pi r h + 2\,\pi r^2$ Find $\frac{dA}{dr}$, then find the minimum value of A

17. A sheet metal worker wishes to make a cylindrical vessel open at one end that will contain one gallon Find the dimensions of the vessel if its total surface is to be a minimum (1 gallon = 231 cu in) $A = \pi r^2 + 2\,\pi r h$

18 Find the shortest path from $B(6, 6)$ to $A(2, 0)$ as shown in the diagram

HINT Let $h = BC + CA$ Find h in terms of k Find the minimum value of h



Instantaneous Velocity [A]

The average velocity for any interval of time is equal to the distance divided by the time If the velocity is constant, the instantaneous velocity is equal to the average velocity If the velocity is not constant but the distance can be expressed as a function of the time, the instantaneous velocity can be found by differentiation

The instantaneous velocity is the limit of $\frac{\Delta s}{\Delta t}$ as Δt approaches zero as a limit This limit is the derivative of s with respect to t

The instantaneous velocity for any time, t, is the time derivative of the distance traversed Thus, if $s = f(t)$, the velocity

$$v = \frac{ds}{dt}$$

Example An object moves in a straight line according to $s = t^2 + 6$ s representing the number of feet and t the number of seconds Find the velocity at the end of 4 seconds Find the distance the object goes in 4 seconds

Solution
$$s = t^2 + 6$$
$$v = \frac{ds}{dt} = 2\,t$$

For $t = 4$, $v = 2 \times 4 = 8$ the velocity in feet per second at the end of 4 seconds

$s = 4^2 + 6 = 22$, the distance the object goes in 4 seconds

EXERCISES

[A]

1 If we neglect friction, a ball dropped from a balloon will fall according to the equation $s = 16.1\,t^2$, s denoting the number of feet it falls and t denoting the number of seconds it falls Find the velocity acquired by the ball in 8 seconds

2 A body moves according to the equation $s = 13\,t - 8\,t^2$, s being expressed in feet and t in seconds Find the velocity at the end of 1 second and interpret your answer Find its greatest distance from the starting point

3 If a body is projected vertically upward with a velocity of 60 feet a second, its distance from the starting point is given by the formula $s = 60\,t - 16.1\,t^2$ Find the distance of the

object above the starting point at the end of 2 seconds. Find the velocity of the object at the end of 3 seconds. When will the velocity of the object first be zero?

4. A ball starts rolling down an inclined plane with a velocity of 6 feet a second. The distance it goes in t seconds is expressed by the formula $s = 6t + 5t^2$. Find the velocity of the ball at the end of 3 seconds.

5. An object moves according to the equation $s = 100t - 8t^2$. Find the velocity of the object at the end of $3\frac{1}{2}$ seconds.

6. A boy fired a rifle vertically upward. If the initial velocity of the bullet was 2700 feet per second, how high did it rise? ($s = 2700t - 16t^2$)

7. A certain motion is expressed by $s = 8 + 60t - 4t^2$, s denoting the distance in feet and t denoting the time in seconds. When will the velocity be 44 feet per second? When will it be zero? Find the maximum value of s.

Checking Your Understanding of Chapter 17

You should be sure that you know

Now check yourself

1. How to find the average rate of change of functions (p. 496)
2. That $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ (p. 499)
3. That a tangent to a curve is the limiting position of a secant as one of its points of intersection with the curve approaches the other as a limit (p. 501)
4. That $\dfrac{dy}{dx} = \underset{\Delta x \to 0}{\text{limit}} \dfrac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{limit}} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ (p. 502)
5. How to find derivatives of certain functions (p. 504)
6. That $D_x y$ is the slope of the tangent to the curve $y = f(x)$ at any point (x, y) on the curve (p. 509)
7. How to find the equation of the tangent to a curve (p. 510)
8. How to find the maximum and minimum values of functions (p. 511)
9. How to find instantaneous velocities (p. 516)

CHAPTER REVIEW

Find the derivatives of the following functions

1 $y = 4x^3$

2 $y = 1 - 10x^2$

3 $s = 40t - 8t^2$

4 $f(x) = (2x - 1)(x + 3)$

5 $y = \frac{4}{x^3}$

6 $y = \frac{x}{\sqrt{x - 1}}$

7 Find the average rate of change of the function $2x^2 - 6x + 3$ in the interval $x = -3$ to $x = 1$

8 Find the slope of the secant of the curve whose equation is $y = x^3 - 6x + 2$ if the secant intersects the curve at the points where $x = 0$ and $x = -3$

9 Show that the graph of $2x^3 - x^2 + x - 1$ has no turning point

For what values of x do the following functions have minimum values?

10 $x^2 - 8x + 3$

11 $x^2 + 5x$

12 $2x^2 - 3x + 2$

13 $5x^2 - 4x + 8$

Find the maximum values of the following functions

14 $10 + x - x^2$

15 $6 - x^2$

16 $6 - 6x - x^2$

17 $1 + 10x - 2x^2$

18 The velocity of a wave in deep water is given by the formula $v = \left(\frac{l}{k} + \frac{k}{l}\right)^{\frac{1}{2}}$ l being the wave length and k a constant Find the wave length that has minimum velocity

19 Find the slope of the curve $y = x^3 - 2x^2 - 6$ at the point where $x = -1$

20 Write the equation of the tangent to the curve $y = x^3 - 4x^2 + 5x - 2$ at the point where $x = 2$

21 Find the turning points of the graph of $y = \frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 1$

22 A body moves from a fixed point according to the equation $s = 2t^2 + 4t + 12$ How far is it from the point in 4 seconds? What is its velocity at the end of 4 seconds?

23 How fast is the area of a square increasing when one of its sides is increasing at the rate of 2 inches per second?

24 When a rifle is fired vertically upward, the initial velocity of the bullet is 2400 feet per second The equation of the motion of the bullet is given by the equation $s = 2400\,t - 16\,t^2$ How long will the bullet rise? How high will the bullet rise?

CHAPTER TESTS

[Test A]

Find the derivatives of the following functions

1. $y = 2x^2$
2. $y = 3x^3 - x$
3. $y = 10 - 4x^2$
4. $s = t^2 - 6t$
5. $f(x) = (x - 4)(x - 1)$
6. $y = x^2 - \frac{1}{x^2}$
7. $y = \sqrt{x^2 - 3}$
8. $y = \frac{4}{x - 1}$

9. Find the average rate of change of the function $x^2 - 3x - 4$ in the interval $x = 1$ to $x = 2$

10 Find the slope of the curve whose equation is $y = x^2 - 5x + 6$ at the point $(x = 1, y = 2)$

11 Find the value of x that makes $y = x^3 - 12x$ a maximum

12 Find the rate of change of the function $y = 2x^2 - 8x$ when $x = 3$

13 A body is thrown vertically upward with an initial velocity of 70 feet per second

How high will it rise if $s = 70\,t - 16\,t^2$?

14. Find the equation of the tangent to the graph of $y = 3x^2 - 15x + 7$ at the point of the curve whose x value is 3

15 If the radius of a circle increases at the rate of 3 feet a second, how fast is its area increasing when the radius is 4 feet?

[Test B]

1 $y = x^3$ Find $\frac{dy}{dx}$

2 Find $D_x\left(\frac{1}{x}\right)$

3. $y = \sqrt{x^2 - 3}$ Find $\frac{dy}{dx}$

4. $y = x^4 - 2x^2 - 6$ Find $\frac{dy}{dx}$

5 The perimeter of an isosceles triangle is 12 inches Show that the triangle has maximum area when its base is 4 inches

6 A sheet-metal worker wishes to make an open gutter of maximum capacity The cross section of the gutter is to be an isosceles trapezoid whose base and sides are each 4 inches Show that the width across the top is 8 inches when the trough has maximum capacity

Mathematics and Research

Within your lifetime a whole new field of job opportunities has appeared—the field of research mathematics No work has ever offered greater challenges and greater opportunities than does the work of this new area, for the research mathematician is at the center of progress He is using his knowledge of mathematics to find and solve problems vital to the welfare of the world

He may be part of a research team which is exploring the secrets of the atom or the workings of man s mind He may be helping to develop a space ship or an automatic defense system to protect the lives of people in case of air attack He may be helping industry to produce fabulous new products—ultrasonic planes, new drugs, new uses for atomic energy

Most research mathematicians are employed in industry, though some are to be found in university research laboratories or in government positions. Usually the mathematician works as a member of a team of scientists, engineers, and technological experts who pool their respective abilities to find solutions for problems This pooling of information is necessary because our knowledge of science has increased so rapidly and to such extent that no one person can be expert in all its phases The same can be said for technology Moreover, instead of dealing with specific parts of total situations, we tend today to develop systems to meet total situations The development of a system demands the thinking of specialists in many areas The research mathematician, therefore, instead of hiding away to solve problems alone in a study, is working with people He is adding his knowledge to the knowledge of experts in many fields

Strange as it may seem, the mathematician may not be solving problems at all His work may be to formulate problems—to sift important information from unimportant, to clarify issues, and to get information in usable form His work may be to suggest appropriate mathematical approaches to problems Of course, some mathematicians do solve problems.

The young person who hopes to become a research mathematician should possess much creative ability and imagination without being impractical He should enjoy research He should be able to work with people He should possess superior ability and a personality that inspires confidence in his opinions.

He should secure broad and thorough training in mathematics, including statistics He should secure broad training in science and other fields in which he is likely to work, and should learn all he can about the applications of mathematics

National Bureau of Standards

CHAPTER 20

Higher-Degree Equations

In this chapter you will learn to solve equations of degree higher than the second.

Number of Roots of a Rational and Integral Equation [A]

So far in our mathematics we have learned how to solve first degree equations in one or more variables, second-degree equations in one variable, and certain systems of second degree equations in two variables Most problems we encounter in the applications of mathematics can be solved by these equations However, in some applications of mathematics, as in chemistry and electricity, we do need to solve equations of degree higher than the second

The solution of higher degree equations is, as we should expect, more difficult than the solution of linear and quadratic equations Fortunately we can solve all higher-degree equations by the same methods, and do not need to learn different methods for equations of different degrees

The general equation of degree n is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \quad + a_n = 0$$

where a_0, a_1, a_2, $\quad$, a_n are constants (real or imaginary), $a_0 \neq 0$, and n is a positive integer

It is possible to derive formulas for finding the roots of all third- and fourth-degree equations, but the formulas are complicated and it is easier to find the roots by other methods Niels Henrik Abel (1802–1829), a Norwegian mathematician, has proved it impossible to derive a formula for the roots of a general algebraic equation of degree higher than four

Theorem Every integral rational equation $f(x) = 0$ has at least one root

The proof of this theorem involves mathematics too advanced for this course, and we shall assume it without proof

It is far from obvious, in fact it is surprising, that every equation of the type

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \quad + a_n = 0$$

has a root, and, further, that it has the same number of roots as it has degree This latter statement we shall prove on page 529

Before proceeding to the solution of equations, we shall study some topics that are very helpful in saving time and making the solutions easier

Remainder Theorem If $f(x)$ is divided by $x - r$ until a remainder is obtained which does not involve x this remainder equals $f(r)$

PROOF If we let $Q(x)$ represent the quotient and R the remainder obtained when a polynomial $f(x)$ is divided by $x - r$, we have

$$f(x) \div (x - r) = Q(x) + \frac{R}{x - r}$$

$\mathrm{M}_{(x-r)}$ $$f(x) = (x - r) \cdot Q(x) + R$$

Since this statement is true for any value of x, we may let $x = r$ We then obtain

$$f(r) = (r - r) \cdot Q(r) + R$$

Then $$f(r) = 0 \cdot Q(r) + R$$
$$f(r) = R$$

Example Find the remainder when $2x^3 - 3x^2 - x + 3$ is divided by $x - 2$

Solution $f(r) = f(2) = 2(2)^3 - 3(2)^2 - (2) + 3$
$= 16 - 12 - 2 + 3 = 5$

Therefore the remainder is 5

Factor Theorem If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of $f(x)$

PROOF $f(x) = (x - r)Q(x) + f(r)$ by the remainder theorem Since r is a root of $f(x) = 0$, then $f(r) = 0$ Substituting 0 for $f(r)$ in the equation, we have $f(x) = (x - r)Q(x)$

Converse of Factor Theorem If $x - r$ is a factor of $f(x)$ then r is a root of the equation $f(x) = 0$

PROOF Since $x - r$ is a factor of $f(x)$,

$$f(x) = (x - r)\, Q(x)$$

If we substitute $x = r$, we have

$$f(r) = (r - r)Q(r) = 0$$

Therefore r is a root of the equation $f(x) = 0$

Synthetic Division[A]

Synthetic division is an abbreviated way of dividing a polynomial by a binomial

Let us divide $2x^3 - 9x^2 + 20$ by $x - 3$ First we shall divide in the usual way, being careful to arrange both the dividend and the

divisor in descending powers of x. We write $0x$ in the proper place so that no power of x is missing

$$
\begin{array}{rl}
2x^3 - 9x^2 + 0x + 20 & \underline{|\,x - 3\qquad\quad} \quad \text{Divisor} \\
\underline{2x^3 - 6x^2} \qquad\qquad & |\,2x^2 - 3x - 9 \quad \text{Quotient} \\
-3x^2 + 0x \qquad\quad & \\
\underline{-3x^2 + 9x} \qquad\quad & \\
-9x + 20 & \\
\underline{-9x + 27} & \\
-\ \ 7 & \text{Remainder}
\end{array}
$$

We can simplify the division by writing only the coefficients

$$
\begin{array}{rl}
2 - 9 + 0 + 20 & \underline{|\,1 - 3\quad} \\
\underline{2 - 6} \qquad\qquad & |\,2 - 3 - 9 \\
-3 + 0 \qquad & \\
\underline{-3 + 9} \qquad & \\
-9 + 20 & \\
\underline{-9 + 27} & \\
-\ \ 7 &
\end{array}
$$

We can simplify the division even more by omitting (1) the first term of each subtrahend since these terms are only repetitions of the numbers above them, (2) the first term of the divisor since it only produces the numbers just omitted, (3) the second term of each difference since these numbers appear in the original polynomial, and (4) the terms of the quotient since these are repetitions of the first terms of the subtrahends. We now have

$$
\begin{array}{rl}
2 - 9 + 0 + 20 & \underline{|\,-3} \\
\underline{-6} \qquad\qquad\quad & \\
-3 \qquad\qquad\quad & \\
\underline{+9} \qquad\quad & \\
-9 \qquad\quad & \\
\underline{+27} & \\
-\ \ 7 &
\end{array}
$$

Now we see that we can make the form more compact by writing it as follows

$$
\begin{array}{rl}
2 - 9 + 0 + 20 & \underline{|\,-3} \\
\underline{-6 + 9 + 27} & \\
2 - 3 - 9 - \ \ 7 &
\end{array}
$$

By substituting $+3$ for -3 in the divisor we are able to add instead of subtracting

$$\begin{array}{l|l} 2-9+0+20 & +3 \\ \underline{\;\;+6-9-27} & \\ 2-3-9-\;\;7 & \end{array}$$

If we now place the proper powers of x to the right of the terms of the third line, we have the quotient $2x^2-3x-9$ and a remainder of -7

To Divide by Synthetic Division

1 Write the coefficients of the dividend in order of the descending powers of x, supplying a zero for each missing power
2 If $x-r$ is the divisor, write r for the indicated divisor. If $x+r$ is the divisor write $-r$ for the indicated divisor
3 Write the first coefficient of the dividend in the first place in the third line
4 Multiply the first coefficient by r and add the product to the second coefficient
5 Multiply this sum by r and add the product to the next coefficient, and so on, until the division is completed
6 The last number in the third line is the remainder. The other numbers in the third line are the coefficients of the quotient in order of descending powers of x, from left to right. The highest power of x in the quotient is of one less degree than that in the dividend

Example Divide x^3-8x+3 by $x+3$

Solution

$$\begin{array}{l|l} 1-0-8+3 & -3 \\ \underline{\;\;-3+9-3} & \\ 1-3+1+0 & \end{array}$$

The quotient is x^2-3x+1 and the remainder is zero

EXERCISES [A]

Divide by synthetic division

1 $(x^3-4x^2+4x-1)\div(x-1)$
2 $(2x^3+3x^2-4x-5)\div(x+1)$
3 $(x^4-3x^3+3x^2-3x+6)\div(x-2)$
4 $(2x^4+6x^3+7x^2+5x-6)\div(x+2)$
5 $(x^4-4x^3-x^2-12)\div(x-4)$

6 $(x^3 - 3x + 5) \div (x - 3)$

7 $(x^5 - 8x^3 + 24x^2 + 12x + 40) \div (x + 4)$

8 $(2y^4 - 13y^3 + 16y^2 - 3y - 10) \div (y - 5)$

9 Given $f(x) = 2x^3 - 4x^2 + 5x - 8$. Use the remainder theorem and synthetic division to find $f(2)$, $f(1)$, $f(-2)$.

10 Given $f(x) = 4x^3 - 7x^2 + 2x - 8$, find $f(2)$, $f(1)$, $f(-1)$, $f(0)$.

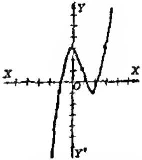
Graphical Solution of Higher-Degree Equations[A]

Approximations of the real roots of a rational integral equation can be found graphically.

Example Find the real roots of $x^3 - 3x^2 + 3 = 0$ graphically.

Solution Set $y = f(x) = x^3 - 3x^2 + 3$.
By synthetic division we obtain the values given in the table.

x	$y = f(x)$
-2	-17
-1	-1
0	3
1	1
2	-1
3	3



The graph of the equation is shown in the figure above. The points where the graph intersects the x axis are the values of x for which $f(x) = 0$. Therefore these values of x are the roots of the equation: $x = -0.9$, 1.3, and 2.5 approximately. The colored arrows in the table indicate where $f(x)$ changes from negative values to positive values or from positive values to negative values. The curve intersects the x axis at least once between any two values of x for which $f(x)$ changes sign.

Be sure that you remember the following principle which we shall use later in this chapter.

Number of Roots of an Algebraic Equation [A]

Theorem An integral rational equation of degree n, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$ $a_0 \neq 0$, has exactly n roots

PROOF By the theorem on page 523 the equation has at least one root Let this root be r_1

By the factor theorem, $x - r_1$ is a factor of the left member of the equation Then the equation becomes

$$(x - r_1)f_1(x) = 0$$

where $f_1(x)$ is a factor of degree $(n - 1)$

But $f_1(x) = 0$ has at least one root Let this root be r_2

Then $x - r_2$ is a factor of $f_1(x)$, and we have

$$(x - r_1)(x - r_2)f_2(x) = 0$$

where $f_2(x)$ is a factor of degree $(n - 2)$

As we continue this process, our equation becomes

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n)f_n(x) = 0$$

where $f_n(x)$ is of degree $(n - n)$, or zero, so that $f_n(x) = a_0$

Therefore $f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0$, and the equation has n roots

If $f(x) = 0$ has another root different from any of these, let r be the root

Then $f(r) = a_0(r - r_1)(r - r_2)(r - r_3) \cdots (r - r_n) \neq 0$, since no factor of $f(r)$ is zero Therefore r is not a root of $f(x) = 0$, and $f(x) = 0$ has exactly n roots

Two or more of the n roots of $f(x) = 0$ may be equal If the same root occurs k times, it is called a multiple root, of multiplicity k A root of multiplicity 2 is called a double root and a root of multiplicity 3 is called a triple root

To Multiply the Roots of an Equation by a Constant [A]

The solution of an equation $f(x) = 0$ is often made easier by transforming it into an equation whose roots bear a stated relation to those of the given equation We shall see how to multiply the roots of an equation by a constant

Example 1 Transform $6x^3 + 7x^2 - 9x + 2 = 0$ into an equation with integral coefficients, and having the coefficient of x^3 equal to 1

Solution $\qquad 6x^3 + 7x^2 - 9x + 2 = 0 \qquad (1)$

$D_6 \qquad x^3 + \frac{7}{6}x^2 - \frac{3}{2}x + \frac{1}{3} = 0 \qquad (2)$

If we substitute $x_1 = 6x$ in equation (2), we obtain an equation in x_1 whose roots are 6 times the roots of $f(x) = 0$ and whose coefficients are integral

Since $x_1 = 6x$, or $x = \frac{x_1}{6}$, we substitute $\frac{x_1}{6}$ for x in (2)

Then $$\frac{x_1^3}{216} + \frac{7}{6}\left(\frac{x_1^2}{36}\right) - \frac{3}{2}\left(\frac{x_1}{6}\right) + \frac{1}{3} = 0$$

or $$\frac{x_1^3}{216} + \frac{7x_1^2}{216} - \frac{x_1}{4} + \frac{1}{3} = 0$$

$M_{216} \qquad x_1^3 + 7x_1^2 - 54x_1 + 72 = 0$ the required equation

That is in this equation the roots of those of the original one are multiplied by 6

The equation just written is said to be in p form with integral coefficients When we transform the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

into the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

in which $a_0 = 1$, we say that we have transformed it into p-form Equation (2) above is in p form If the coefficients $p_1, p_2, \cdots$ are all integers, we say that the equation is in p form with integral coefficients Any rational root of an equation in p-form with integral coefficients is an integer and an exact divisor of p_n

Example 2 Transform $x^3 - 3x^2 + 3x - 9 = 0$ into an equation in x_1 whose roots have the same absolute value but are opposite in sign

Solution To obtain an equation whose roots have the same absolute value but are opposite in sign to those of $x^3 - 3x^2 + 3x - 9 = 0$ we must multiply its roots by negative 1

$$x^3 - 3x^2 + 3x - 9 = 0 \qquad (1)$$

Substituting $x = -x_1$, we obtain

$$(-x_1)^3 - 3(-x_1)^2 + 3(-x_1) - 9 = 0 \qquad (2)$$

Simplifying $\qquad -x_1^3 - 3x_1^2 - 3x_1 - 9 = 0 \qquad (3)$

$D_{-1} \qquad x_1^3 + 3x_1^2 + 3x_1 + 9 = 0 \qquad (4)$

From an inspection of equations (3) and (1) in Example 2 we see that we can obtain an equation whose roots are opposite in sign to

those of another equation by changing the signs of the coefficients of the odd degree terms We shall frequently need to do this in solving equations

[A]

Obtain equations whose roots are equal in absolute value but opposite in sign to the roots of the following equations

1 $x^3 - 4x^2 - 4x + 12 = 0$ 3 $x^3 - x^2 + 5 = 0$

2 $x^3 - 3x - 3 = 0$ 4 $x^3 - 4x^2 + 7x - 3 = 0$

5 $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$

Transform the following equations so as to multiply the roots by the smallest integer which will transform the equations into a p form with integral coefficients

6 $x^3 - \frac{3}{4}x^2 - \frac{17}{8}x + \frac{3}{8} = 0$ Let $x = \frac{x_1}{4}$ Why?

7 $x^3 - 3x^2 - \frac{3}{4}x + \frac{5}{4} = 0$ 9 $10x^3 - 13x^2 + 1 = 0$

8 $2x^3 - 3x^2 + 2x - 3 = 0$ 10 $7x^3 + 4x^2 - 17x + 6 = 0$

Depressed Equations [A]

If r is a root of an equation $f(x) = 0$, then $(x - r)Q(x) = 0$ The equation $Q(x) = 0$ is the depressed equation of $f(x)$ The roots of $Q(x) = 0$ are the remaining roots of $f(x) = 0$

Example Find the remaining roots of the equation $x^3 - 3x^2 + 4x - 12 = 0$ if one root is 3

Solution Dividing $x^3 - 3x^2 + 4x - 12$ by $x - 3$, using synthetic division, we have

$$\begin{array}{r|l} 1 - 3 + 4 - 12 & 3 \\ +3 + 0 + 12 & \\ \hline 1 + 0 + 4 + 0 & \end{array}$$

The depressed equation is $x^2 + 4 = 0$ Solving $x^2 + 4 = 0$, we have $x = \pm 2i$ The three roots of the original equation are $3, +2i$, and $-2i$

Descartes's Rule of Signs [A]

If the equation $f(x) = 0$ is arranged in standard form, there is said to be a variation of sign whenever two successive terms have opposite signs Thus the equation $2x^4 - 5x^3 + x^2 - x - 6 = 0$ has three variations of signs From $+2x^4$ to $-5x^3$ is one, from $-5x^3$ to $+x^2$ is another, and from $+x^2$ to $-x$ is the third variation of sign

The following theorem, known as Descartes's rule of signs, was formulated by the French mathematician René Descartes

The number of real positive roots of the equation $f(x)=0$ does not exceed the number of variations of sign in $f(x)=0$

The number of real negative roots of the equation $f(x)=0$ does not exceed the number of variations of sign in $f(-x)=0$

Example 1 Apply Descartes's rule to $x^4+x^2+x+2=0$

Solution 1 Since there are no variations of sign in $f(x)$, there are no positive real roots

2 $f(-x)=x^4+x^2-x+2$ Since there are 2 variations of sign in $f(-x)$, the number of negative real roots does not exceed 2

3 Since an equation of the fourth degree has 4 roots, and from steps 1 and 2 the number of real roots does not exceed 2, there are at least 2 imaginary roots

Algebraic Solution of Higher-Degree Equations [A]

Already in this chapter we have studied the solution of higher-degree equations by graphing Now we shall study procedures for finding solutions algebraically We shall confine our attention to two cases (1) when either all the roots, or all the roots but two, are rational, and (2) when not more than two roots are imaginary Finding all the roots of a rational integral equation when more than two roots are imaginary is beyond the scope of this book

Algebraic Solution of Higher-Degree Equations When All the Roots, or All the Roots but Two, are Rational [A]

Example 1 Solve $x^4-4x^3+x^2+8x-6=0$

Solution. 1 Applying Descartes's rule of signs, $f(x)$ has 3 variations of sign, and so the number of positive real roots of $f(x)=0$ does not exceed 3 $f(-x)=x^4+4x^3+x^2-8x-6$ has 1 variation of sign, and so the number of negative roots does not exceed 1

2 If the equation has rational roots, we know that the possible roots of the equation are integral factors of -6 The factors of -6 are ±1, ±2, ±3, and ±6

3 If we test the possibility of $+1$ as a root by synthetic division we have

$$\begin{array}{r} 1-4+1+8-6\,\lfloor 1 \\ +1-3-2+6 \\ \hline 1-3-2+6+0 \end{array}$$

Since the remainder is zero 1 is a root and the depressed equation is $x^3-3x^2-2x+6=0$ The remaining roots are the roots of this depressed equation

4 If we test the possibility of $+2$ as a root of $x^3-3x^2-2x+6=0$ by synthetic division we have

$$\begin{array}{r} 1-3-2+6\,\lfloor 2 \\ +2-2-8 \\ \hline 1-1-4-2 \end{array}$$

Since the remainder is -2 2 is not a root of the equation

5 If we test the possibility of $+3$ as a root of the equation we have

$$\begin{array}{r} 1-3-2+6\,\lfloor 3 \\ +3+0-6 \\ \hline 1+0-2+0 \end{array}$$

Since the remainder is 0 3 is a root of the equation and the second depressed equation $x^2-2=0$

6 Solving the equation $x^2-2=0$ we obtain $x=\pm\sqrt{2}$

7 Therefore the roots of the equation $x^4-4x^3+x^2+8x-6=0$ are 1 3 $\sqrt{2}$ and $-\sqrt{2}$

Example 2 Solve $2x^3-9x^2+14x-5=0$

Solution 1 $f(x)$ has 3 variations of sign and so the number of positive real roots does not exceed 3 $f(-x)$ has no variations in sign and so there are no negative real roots

2 To transform the equation $2x^3-9x^2+14x-5=0$ into an equation in which the coefficient of x^3 is 1 and in which there are no fractions we must multiply the roots by 2

$$2x^3-9x^2+14x-5=0$$

D_2 $$x^3-\tfrac{9}{2}x^2+7x-\tfrac{5}{2}=0$$

To multiply the roots by 2 we let $x=\frac{x_1}{2}$

Then $$\frac{x_1^3}{8}-\frac{9x_1^2}{8}+\frac{7x_1}{2}-\frac{5}{2}=0$$

and $$x_1^3-9x_1^2+28x_1-20=0$$

3 The possible rational roots of the original equation are either the factors of the constant term or the constant term divided by 2 These are ± 1, ± 5, $\pm \frac{1}{2}$ and $\pm \frac{5}{2}$

Since the roots of $x_1^3 - 9x_1^2 + 28x_1 - 20 = 0$ are twice the roots of the original equation, and from step 1 there are no negative roots, possible roots of the equation in x_1 are 1, 2, 5, and 10

Trying 1 we have

$$\begin{array}{rrrr|l} 1 & -9 & +28 & -20 & \underline{1} \\ & +1 & -8 & +20 & \\ \hline 1 & -8 & +20 & +0 & \end{array}$$

The remainder is zero, so that 1 is a root of $x_1^3 - 9x_1^2 + 28x_1 - 20 = 0$ and the depressed equation is $x_1^2 - 8x_1 + 20 = 0$

Solving $x_1^2 - 8x_1 + 20 = 0$ by the quadratic formula, we have $x_1 = 4 \pm 2i$

4 The roots of $x_1^3 - 9x_1^2 + 28x_1 - 20 = 0$ are 1 and $4 \pm 2i$

Therefore the roots of $2x^3 - 9x^2 + 14x - 5 = 0$, which are one half the roots of $x_1^3 - 9x_1^2 + 28x_1 - 20 = 0$, are $\frac{1}{2}$ and $2 \pm i$

From the above examples we can formulate the following rules to solve equations when all the roots, or all the roots but two, are rational

1 Determine the possibility of positive and negative real roots by Descartes's rule of signs.
2 If the equation is not in the p form, transform it into the p form with integral coefficients.
3 Determine possible absolute values of the roots
4 Depress this equation by finding all but two of the roots.
5 Solve the resulting quadratic equation for the two remaining roots.
6 If the original equation was not in the p form, divide the roots by the proper constant to obtain the roots of the original equation

In some cases the left member of $f(x) = 0$ may be factored directly, as is illustrated in the examples that follow

Example 3 Solve $x^3 + 1 = 0$

Solution In this equation it is easier first to factor the left member

$$x^3 + 1 = 0$$
$$(x + 1)(x^2 - x + 1) = 0$$

Then $\quad x+1=0$ and $x^2-x+1=0$

$$x=-1 \text{ and } x=\frac{1\pm\sqrt{1-4}}{2}, \text{ or } \frac{1\pm i\sqrt{3}}{2}$$

Example 4 Solve $x^3-x=0$

Solution. Factoring, $x(x^2+1)(x^2-1)=0$
Then $x=0$, $x^2+1=0$ and $x^2-1=0$
Solving the quadratics, $x=\pm i$ and $x=\pm 1$
The roots are 0, $\pm i$, and ± 1

EXERCISES [A]

Solve

1. $x^4-1=0$
2. $x^6-1=0$
3. $x^4-x=0$
4. $x^6-64=0$
5. $6x^3+x^2-2x=0$
6. $x^5-16x=0$
7. $x^3-8x^2+8x+8=0$
8. $x^4-4x^3+x^2+6x=0$
9. $x^3-3x-2=0$
10. $x^3-3x+2=0$
11. $6x^3-11x^2-3x+2=0$
12. $5x^3-29x^2+21x-5=0$
13. $6x^4+x^3-19x^2+6x=0$
14. $x^4-4x^2+3x+6=0$
15. $x^3+5x^2+2x+10=0$
16. $x^4+x^3-10x^2-12x=0$
17. $x^4-5x^3-11x^2-19x-14=0$

Upper and Lower Limits for Roots [A]

Because it narrows our search, it is sometimes helpful in finding the roots of an equation, to know that they lie within a certain interval If $f(x)$ has no real root greater than the integer a, then a is an upper limit for the real roots Likewise, if b is less than any real root, b is a lower limit of the real roots

To find an upper limit a for the positive roots of $f(x)=0$ we use the following theorem

When we divide $f(x)$ by $x-a$ by synthetic division, a being positive or zero, if all the numbers in the third line of the synthetic division are zero or positive, then a is an upper limit of the positive roots of $f(x)=0$

To find the lower limit b for the negative roots of $f(x) = 0$ it is only necessary to find an upper limit for the positive roots of $f(-x) = 0$

Example Find an upper and a lower limit of the real roots of $x^3 - 3x^2 - 4x + 10 = 0$

Solution To find an upper limit of the positive roots we pick a number at random If we try 3 as a possibility, we have

$$\begin{array}{rrrr|l} 1 & -3 & -4 & +10 & 3 \\ & +3 & +0 & -12 & \\ \hline 1 & +0 & -4 & -2 & \end{array}$$

Since not all the numbers in the third line are positive or zero, 3 is not an upper limit of the real roots If we try 4, we have

$$\begin{array}{rrrr|l} 1 & -3 & -4 & +10 & 4 \\ & +4 & +4 & +0 & \\ \hline 1 & +1 & +0 & +10 & \end{array}$$

Since the numbers in the third line are all positive or zero, 4 is an upper limit of the roots of $x^3 - 3x^2 - 4x + 10 = 0$

To find a lower limit of the negative roots of the equation we find $f(-x)$ by changing the signs of the coefficients of the odd-degree terms Doing this, we get $-x^3 - 3x^2 + 4x + 10 = 0$ Dividing both members of the equation by -1, we get $x^3 + 3x^2 - 4x - 10 = 0$ Then $f(-x) = x^3 + 3x^2 - 4x - 10$ Again we pick a number at random If we try 1 as an upper limit for the positive roots of $x^3 + 3x^2 - 4x - 10 = 0$, we have

$$\begin{array}{rrrr|l} 1 & +3 & -4 & -10 & 1 \\ & +1 & +4 & -0 & \\ \hline 1 & +4 & +0 & -10 & \end{array}$$

Since 1 does not make all the numbers in the third line positive or zero we try 2

$$\begin{array}{rrrr|l} 1 & +3 & -4 & -10 & 2 \\ & +2 & +10 & +12 & \\ \hline 1 & +5 & +6 & +2 & \end{array}$$

Then 2 is an upper limit of the positive roots of $f(-x) = 0$ and hence -2 is a lower limit of the negative roots of $f(x) = 0$

In finding the real roots of $f(x) = 0$ it is customary to find the positive roots of $f(x) = 0$ and the positive roots of $f(-x) = 0$

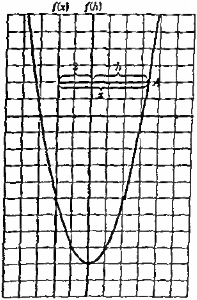
EXERCISES [A]

Find upper and lower limits for the real roots of the following equations

1. $2x^3 - 5x^2 + 3 = 0$
2. $3x^4 - 5x^2 + x - 8 = 0$
3. $3x^3 + x^2 - 9x + 5 = 0$
4. $x^5 - 3x^3 - 16 = 0$
5. $x^4 - 3x^3 - x^2 + 6 = 0$
6. $x^5 - 3x^3 + 5x + 10 = 0$
7. $x^3 - 3x^2 - 6x - 12 = 0$
8. $x^4 - 4x^3 - x + 8 = 0$

To Diminish the Roots of an Equation [8]

When it is necessary to find the irrational roots of a higher-degree equation, we find that the computation is simpler if we can diminish the roots of the equation by an integral or decimal value



Let us consider the equation $x^2 - 4x - 5 = 0$ whose graph is shown at the right The roots of the equation are $+5$ and -1

If we decrease the roots by 2, they become $+3$ and -3, respectively Study of the graph shows that this is equivalent to shifting the $f(x)$ axis 2 units to the right

Let us now see what effect the change has on the equation Notice that whereas point A on the curve is x units to the right of the original $f(x)$ axis, it is $x - 2$ units to the right of the new $f(h)$ axis If we let $x = h + 2$ in the equation $x^2 - 4x - 5 = 0$, we obtain $(h+2)^2 - 4(h+2) - 5 = 0$ or $h^2 - 9 = 0$ The roots of $h^2 - 9 = 0$ are $+3$ and -3 as we expected

We can also obtain the equation $h^2 - 9 = 0$ by a series of divisions Since $h = x - 2$, we shall divide $x^2 - 4x - 5$ by $x - 2$, obtaining a quotient plus a remainder We shall then divide the quotient (we do not include the remainder) by $x - 2$ getting a new quotient and a new remainder This last quotient will show the number of times $(x-2)(x-2)$ or $(x-2)^2$ appears in $x^2 - 4x - 5 = 0$, and its re-

mainder tells how many times $(x-2)^1$ appears The first remainder is the constant, that is, the number of times $(x-2)^0$ appears Since $(x-2)^2 = h^2$ and $x - 2 = h$, we can then write the equation in h Actually performing the work by synthetic division we have

$$\begin{array}{l}
1 - 4 - 5 \;\underline{|2} \\
\underline{\;\;+2 - 4} \\
1 - 2 \,|\! - 9 \\
\underline{\;\;+2\,|} \\
1 + 0
\end{array}$$

Do you recognize that the numbers in color are the coefficients of the transformed equation $h^2 - 0\,h - 9 = 0$ as explained above?

While either of the methods mentioned can be used to transform a given equation into a new equation each of whose roots is less by c than the corresponding root of the given equation, the method of repeated divisions is better for equations of high degree Restated in general terms, the rule for repeated divisions is

To obtain an equation each of whose roots is less by c than the corresponding root of a given equation $f(x) = 0$ of degree n, divide $f(x)$ by $x - c$ and indicate the remainder by R_n Divide the quotient by $x - c$ and indicate the remainder by R_{n-1} Continue to divide each quotient by $x - c$ until n divisions have been performed The last quotient, a_0, and the remainders, R_1, R_2, , R_n, are the coefficients of the terms of the transformed equation

$$a_0h^n + R_1h^{n-1} + R_2h^{n-2} + \quad + R_{n-1}h + R_n = 0$$

Example 1 Determine the equation whose roots are 2 less than those of $2x^3 - 5x^2 + 3 = 0$

Solution Divide $2x^3 - 5x^2 + 3$ and each resulting quotient by $x - 2$ until three successive divisions have been performed, 3 being the degree of the equation

$$\begin{array}{ll}
2 - 5 - 0 + 3 \;\underline{|2} & \\
\underline{\;\;+4 - 2 - 4} & \\
2 - 1 - 2 \,|\! - 1 & R_3 = -1 \\
\underline{\;\;+4 + 6\,|} & \\
2 + 3 \,|\, + 4 & R_2 = +4 \\
\underline{\;\;+4\,|} & \\
2 + 7 & R_1 = +7 \\
 & a_0 = +2
\end{array}$$

The successive remainders are the coefficients of the required equation, as shown by a_0, R_1, etc. Hence the required equation is $2y^3 + 7y^2 + 4y - 1 = 0$

Example 2 Diminish the roots of $x^3 + 9x^2 + 18x - 20 = 0$ by 0.7.

Solution.

$$
\begin{array}{llll|l}
1 + & 9 & +18 & -20 & \underline{|0.7} \\
+ & 0.7 + & 6.79 & +17.353 & \\
\hline
1 + & 9.7 + & 24.79 & -2.647 & R_3 = -2.647 \\
+ & 0.7 + & 7.28 & & \\
\hline
1 + & 10.4 & +32.07 & & R_2 = 32.07 \\
+ & 0.7 & & & \\
\hline
1 + & 11.1 & & & R_1 = 11.1 \\
 & & & & a_0 = 1
\end{array}
$$

The required equation is $y^3 + 11.1y^2 + 32.07y - 2.647 = 0$

[9] **EXERCISES**

Find equations whose roots are the roots of the given equations diminished by h

1 $x^3 - 3x^2 - 6x - 12 = 0$	$h = 2$
2 $3x^4 - 5x^2 + x - 8 = 0$	$h = 1$
3 $x^5 - 3x^3 - 16 = 0$	$h = 2$
4 $x^4 - 3x^3 - x^2 + 6 = 0$	$h = 3$
5 $x^4 + 8x^3 + 18x^2 - x - 22 = 0$	$h = 0.9$
6 $x^4 - x^2 - x + 1 = 0$	$h = 0.5$

Algebraic Solution of Higher-Degree Equations When Not More than Two Roots Are Imaginary* [5]

A method of approximating the values of the roots of a higher-degree equation was developed by Horner (1786–1837), an Englishman. The method will be illustrated by an example

Example 1 Solve $x^3 + x^2 - 3 = 0$, finding the real roots to two decimal places

Solution. By Descartes's rule of signs there may be 1 positive root. We then find by trial the upper limit of this possible root

*There is a method known as Graeffe's method for finding all the roots (real and imaginary) of a rational integral equation

and divide by integral values until we obtain a change of sign in the remainders

$$\begin{array}{rrrr|l} 1+ & 1+ & 0- & 3 & \underline{2} \\ & +2+ & 6+ & 12 & \\ \hline 1+ & 3+ & 6+ & 9 & \end{array} \quad \text{Then 2 is an upper limit.}$$

$$\begin{array}{rrrr|l} 1+ & 1+ & 0- & 3 & \underline{1} \\ & +1+ & 2+ & 2 & \\ \hline 1+ & 2+ & 2- & 1 & \end{array}$$

Since the remainders change sign between $x = 2$ and $x = 1$, there is a root between 1 and 2. We therefore transform the equation into an equation whose roots are 1 less than those of the given equation.

$$\begin{array}{rrrr|l} 1+ & 1+ & 0- & 3 & \underline{1} \\ & +1+ & 2+ & 2 & \\ \hline 1+ & 2+ & 2 & -1 & \\ & +1+ & 3 & & \\ \hline 1+ & 3 & +5 & & \\ & +1 & & & \\ \hline 1+ & 4 & & & \end{array}$$

$y^3 + 4y^2 + 5y - 1 = 0$ is the required equation.

We find an approximation to the root of this equation by setting $5y - 1 = 0$. Then $y = 0.2$. Dividing by $(y - .2)$ by synthetic division, we have

$$\begin{array}{rrrr|l} 1+ & 4 & +5 & -1 & \underline{.2} \\ & +.2+ & .84+ & 1.168 & \\ \hline 1+ & 4.2+ & 5.84+ & .168 & \end{array}$$

Since all the numbers in the third line are positive, .2 is an upper limit of the positive root. Dividing by .1, we have

$$\begin{array}{rrrr|l} 1+ & 4 & +5 & -1 & \underline{.1} \\ & +.1+ & .41+ & .541 & \\ \hline 1+ & 4.1+ & 5.41- & .459 & \end{array}$$

Since the remainders change sign between .1 and .2, there is a root of this equation between .1 and .2.

Transforming $y^3 + 4y^2 + 5y - 1 = 0$ into an equation $f(z) = 0$ whose roots are .1 less, we obtain

$$
\begin{array}{l}
1 + 4 \quad + 5 \quad - 1 \quad \underline{|1} \\
\underline{\quad + \; 1 + \; 41 + \; 541} \\
1 + 4 1 + 5 41 \,|\, - \; 459 \\
\underline{\quad + \; 1 + \; 42} \\
1 + 4 2 \,|\, + 5 83 \\
\underline{\quad + \; 1} \\
1 + 4 3
\end{array}
$$

$$z^3 + 4\ 3\,z^2 + 5\ 83\,z - 459 = 0$$

Solving $5\ 83\,z - 459 = 0$ for an approximate value of z, we obtain $z = 07$

Then dividing $f(z)$ by 07, we have

$$
\begin{array}{l}
1 + 4 3 \quad + 5 83 \quad - 459 \quad \underline{|07} \\
\underline{\quad + \; 07 + \; 3059 + \; 429513} \\
1 + 4\ 37 + 6\ 1359 - \; 029487
\end{array}
$$

Dividing by 08, we have

$$
\begin{array}{l}
1 + 4 3 \quad + 5 83 \quad - 459 \quad \underline{|08} \\
\underline{\quad + \; 08 + \; .3504 + \; 494432} \\
1 + 4\ 38 + 6\ 1804 + \; 035432
\end{array}
$$

Since there is a change of sign in the remainders between 07 and 08, a root of $f(z) = 0$ lies between 07 and 08

Therefore the positive root of $x^3 + x^2 - 3 = 0$ to 2 decimal places is $1 + 1 + 07$, or $1\ 17$

Removing the root $1\ 17$ from $x^3 + x^2 - 3 = 0$,

$$
\begin{array}{l}
1 + 1 \quad + 0 \quad\quad - 3 \quad\quad \underline{|1\ 17} \\
\underline{\quad + 1\ 17 + 2\ 5389 + 2\ 970513} \\
1 + 2\ 17 + 2\ 5389 - 0\ 029487
\end{array}
$$

This gives $x^2 + 2\ 17\,x + 2\ 5389 = 0$ as the depressed equation The remainder would have been zero if $1\ 17$ had been an exact value

Solving $x^2 + 2\ 17\,x + 2\ 5389 = 0$ by the quadratic formula,

$$x = \frac{-2\ 17 \pm \sqrt{4\ 7089 - 10\ 1556}}{2}$$

$$x = \frac{-2\ 17 \pm 2\ 33\,i}{2} = -1\ 09 \pm 1\ 17\,i, \text{ approximately}$$

for the other two roots of $x^3 + x^2 - 3 = 0$

To obtain the imaginary roots accurate to 2 decimal places it would have been necessary to find the real root to 3 decimal places

Example 2 Solve $x^4 + x^3 - 12x^2 + 3x + 12 = 0$, finding the roots to 2 decimal places.

Solution There are 2 variations of sign in $f(x)$, so that the number of positive roots does not exceed 2. $f(-x) = x^4 - x^3 - 12x^2 - 3x + 12$ has 2 variations in sign, so that the number of negative roots does not exceed 2.

Locating the positive roots of $f(x) = 0$ by synthetic division,

$$\begin{array}{l|l} 1 + 1 - 12 + 3 + 12 & \underline{3} \\ \underline{ + 3 + 12 + 0 + 9} & \\ 1 + 4 + 0 + 3 + 21 & \end{array} \qquad \text{The upper limit}$$

A root between 2 and 3

$$\begin{array}{l|l} 1 + 1 - 12 + 3 + 12 & \underline{2} \\ \underline{ + 2 + 6 - 12 - 18} & \\ 1 + 3 - 6 - 9 - 6 & \end{array}$$

A root between 1 and 2

$$\begin{array}{l|l} 1 + 1 - 12 + 3 + 12 & \underline{1} \\ \underline{ + 1 + 2 - 10 - 7} & \\ 1 + 2 - 10 - 7 + 5 & \end{array}$$

$$\begin{array}{l|l} 1 + 1 - 12 + 3 + 12 & \underline{0} \\ \underline{ + 0 + 0 + 0 + 0} & \\ 1 + 1 - 12 + 3 + 12 & \end{array}$$

Locating the positive roots of $f(-x) = 0 = x^4 - x^3 - 12x^2 - 3x + 12$, we have

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{5} \\ \underline{ + 5 + 20 + 40 + 185} & \\ 1 + 4 + 8 + 37 + 197 & \end{array} \qquad \text{An upper limit}$$

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{4} \\ \underline{ + 4 + 12 + 0 - 12} & \\ 1 + 3 + 0 - 3 + 0 & \end{array} \qquad \text{A rational root, since the remainder is zero}$$

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{3} \\ \underline{ + 3 + 6 - 18 - 63} & \\ 1 + 2 - 6 - 21 - 51 & \end{array}$$

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{2} \\ \underline{ + 2 + 2 - 20 - 46} & \\ 1 + 1 - 10 - 23 - 34 & \end{array}$$

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{1} \\ \underline{ + 1 + 0 - 12 - 15} & \\ 1 + 0 - 12 - 15 - 3 & \end{array} \qquad \text{A root between 1 and 0}$$

$$\begin{array}{l|l} 1 - 1 - 12 - 3 + 12 & \underline{0} \\ \underline{ + 0 + 0 + 0 + 0} & \\ 1 - 1 - 12 - 3 + 12 & \end{array}$$

Since 4 is a rational root of $f(-x)=0$, -4 is a root of $f(x)=0$

Removing the root -4 from $f(x)=0$, we have

$$\begin{array}{l} 1+1-12+3+12\,\underline{|-4} \\ \underline{\;\;-4+12+0-12} \\ 1-3+\;\;0+3+\;\;0 \end{array}$$

and $x^3-3x^2+3=0$ is the depressed equation. Then we can solve $x^3-3x^2+3=0$ for the remaining roots of the given equation

Suppose we find the positive root of $x^3-3x^2+3=0$ between 1 and 2. First we find an equation $f(y)\approx 0$ whose roots are 1 less than the roots of $x^3-3x^2+3=0$

$$\begin{array}{l} 1-3+0+3\,\underline{|1} \\ \underline{\;\;+1-2-2} \\ 1-2-2\,|+1 \\ \underline{\;\;+1-1} \\ 1-1\,|-3 \\ \underline{\;\;+1} \\ 1+0 \end{array}$$

Then $y^3-3y+1=0$ is the required equation. If we set $-3y+1=0$, we obtain $y=3$ as an approximation

$$\begin{array}{l} 1+0\;\;-3\;\;\;+1\;\;\;\underline{|3} \\ \underline{\;\;+\;\;3+\;\;09-\;\;873} \\ 1+\;\;3-2\,91+0\,127 \end{array}$$ A root between 3 and 4

$$\begin{array}{l} 1+0\;\;-3\;\;\;+1\;\;\;\underline{|4} \\ \underline{\;\;+\;\;4+\;\;16-1\,136} \\ 1+\;\;4-2\,84-\;.136 \end{array}$$

An equation $f(z)=0$ whose roots are 3 less than $f(y)=0$ is

$$\begin{array}{l} 1+0\;\;-3\;\;\;+1\;\;\;\underline{|3} \\ \underline{\;\;+\;\;3+\;\;09-\;\;873} \\ 1+\;\;3-2\,91\,|+0\,127 \\ \underline{\;\;+\;\;3+\;\;18} \\ 1+\;\;6\,|-2\,73 \\ \underline{\;\;+\;\;3} \\ 1+\;\;9 \end{array}$$

$z^3+9z^2-2\,73\,z+127=0$

Solving $-2\,73\,z+127=0$, we obtain $z=04$ as an approximation

$$\begin{array}{l} 1 + 9 \;-2\,73 \;+127 \quad \lfloor 04 \\ \underline{\;+04 + \;0376 - \;107696} \\ 1 + 94 - 2\,6924 + 019304 \end{array}$$

$$\begin{array}{l} 1 + 9 \;-2\,73 \;+127 \quad \lfloor 05 \\ \underline{\;+05 + \;0475 - \;134125} \\ 1 + 95 - 2\,6825 - 007125 \end{array}$$

The root of $f(z) = 0$ lies between 04 and 05 Therefore a root of $x^3 - 3x^2 + 3 = 0$ is $1 + 3 + 04 = 1\,34$

Proceeding to find the root of $x^3 - 3x^2 + 3 = 0$ between 2 and 3, we have

$$\begin{array}{l} 1 - 3 + 0 + 3 \,\lfloor 2 \\ \underline{\;+2 - 2 - 4} \\ 1 - 1 - 2 \,|\, -1 \\ \underline{\;+2 + 2} \\ 1 + 1 \,|\, +0 \\ \underline{\;+2} \\ 1 + 3 \end{array}$$

Then $y^3 + 3y^2 - 1 = 0$ is the equation whose roots are 2 less than $x^3 - 3x^2 + 3 = 0$

$$\begin{array}{l} 1 + 3 \;+0 \;\;-1 \quad \lfloor 6 \\ \underline{\;+\;6 + 2\,16 + 1\,296} \\ 1 + 3\,6 + 2\,16 + \;296 \end{array}$$

A root lies between 5 and 6

$$\begin{array}{l} 1 + 3 \;+0 \;\;-1 \quad \lfloor 5 \\ \underline{\;+\;5 + 1\,75 + \;875} \\ 1 + 3\,5 + 1\,75 \,|\, - \;125 \\ \underline{\;+\;5 + 2} \\ 1 + 4 \,|\, + 3\,75 \\ \underline{\;+\;5} \\ 1 + 4\,5 \end{array}$$

Then $z^3 + 4\,5\,z^2 + 3\,75\,z - 125 = 0$ is the equation whose roots are 5 less than those of $f(y) = 0$

$$\begin{array}{l} 1 + 4\,5 \;+3\,75 \;\;-125 \quad \lfloor 03 \\ \underline{\;+\;03 + \;1359 + \;116577} \\ 1 + 4\,53 + 3\,8859 - 008423 \end{array}$$

$$\begin{array}{l} 1 + 4\,5 \;+3\,75 \;\;-125 \quad \lfloor 04 \\ \underline{\;+\;04 + \;1816 + \;157264} \\ 1 + 4\,54 + 3\,9316 + 032264 \end{array}$$

Solve

5 $x^3 - x - 1 = 0$

6 $x^3 + 5x^2 - 3 = 0$

7 $x^3 + 6x^2 + 8x - 8 = 0$

8 $x^3 - 2x^2 - 3x + 3 = 0$

9 $x^3 - 3x^2 - 4x + 13 = 0$

10 $x^3 + 2x^2 - 6 = 0$

11 $x^4 + x^3 - 6x^2 - 4x + 1 = 0$

12. $x^4 + 4x^3 - 8x^2 + 2 = 0$

13 $2x^3 - 6x^2 + 6x - 3 = 0$

14 $x^4 - 3x^3 + x + 3 = 0$

Solve the following sets of quadratic equations

15 $x^2 + y^2 = 5$
$xy + 2y = 4$

16 $x^2 + y = 10$
$x + y^2 = 4$

17 $x^2 - y = 2$
$y^2 - x = 2$

18 $x^2 + y^2 - x = 10$
$x^2 - x + y = 8$

Checking Your Understanding of Chapter 20

To check your understanding of Chapter 20

Should you review

1 Make sure that you understand clearly that

a Every integral rational equation $f(x) = 0$ has at least one root (p 523)

b An integral rational equation of degree n has exactly n roots (p 529)

2 Make sure that you understand and can use the various helps for solving equations These include

a The remainder theorem (p 524)

b The factor theorem (p 524)

c Synthetic division (p 524)

d The forming of an equation whose roots are a certain number of times those of a given equation (p 529)

e Descartes's rule of signs (p 531)

f Procedure for finding the upper and lower limits of roots (p 535)

g If you expect to continue the study of mathematics, you will also want to be sure you can use the procedure for diminishing the roots of an equation (p 537)

3 Check your ability to solve higher-degree equations by graphing (p 527)

4 Check your ability to solve higher-degree equations algebraically when all the roots, or all the roots but two, are rational (p 532) *If you expect to continue the study of mathematics, you will also want to be sure you can use Horner's method to solve higher-degree equations when not more than two roots are imaginary* (p 539)

CHAPTER TESTS

(Test A)

1. Divide $x^4 - x^3 - 8x^2 + 9x - 9$ by $x - 3$, using synthetic division

2. How many possible positive roots and how many possible negative roots does the equation $x^4 - 2x^3 + 5x^2 - 7 = 0$ have, according to Descartes's rule of signs?

3. Transform $2x^3 - x^2 + 4x + 2 = 0$ into an equation whose roots are 3 times as great

4. Find the depressed equation of $x^4 + x^3 - 7x^2 - x + 6 = 0$ if the root 2 is removed

5. Solve $x^4 + x^3 - 5x^2 - 3x + 6 = 0$

(Test B)

1 Find the positive root of $x^3 + x^2 - 4 = 0$ to 2 decimal places

2. Find the negative root of $x^3 - x + 5 = 0$ to 2 decimal places

CHAPTER

21

Mathematical Induction

In this chapter
a new kind of proof
will be used

Mathematical Induction [A]

This is a very important method of proof used to prove theorems in mathematics In learning the method we shall confine our work to simple series dealing with positive integers A proof by mathematical induction consists of two parts

1 Proof that the theorem or principle in question is true for a particular case—say for $n = k$

2 Proof that if it is true for a particular case ($n = k$), it is true for $n = k + 1$

The method of proof will be illustrated by the following theorem

Example 1 Prove that $1 + 2 + 3 + \quad + n = \frac{n}{2}(n + 1)$,

where n is any positive integer

PROOF *Part I* Verify the theorem for a few integral values of n

If $n = 1$, the left member reduces to the single term 1, and the right member equals $\frac{1}{2}(1 + 1)$ or 1

If $n = 2$, the left member reduces to $1 + 2$, which equals 3, and the right member equals $\frac{2}{2}(2 + 1)$, or 3

Part II Suppose that the theorem is valid for $n = k$, where k is any positive integer

Then $$1 + 2 + 3 + \quad + k = \frac{k}{2}(k + 1) \tag{1}$$

We proceed to prove that the theorem is valid for $n = k + 1$ if it is valid for $n = k$

When $n = k + 1$, we have

$$1 + 2 + 3 + \quad + k + k + 1 = \frac{k + 1}{2}(k + 1 + 1) \tag{2}$$

Adding $k + 1$ to (1), we have

$$1 + 2 + 3 + \quad + k + k + 1 = \frac{k}{2}(k + 1) + (k + 1) \tag{3}$$

The left members of (2) and (3) are identical If we can prove that the right members of (2) and (3) are identical, we shall have proved that the theorem is valid for $n = k + 1$, provided that it is valid for $n = k$

If we simplify the right member of (2), we have $\frac{k + 1}{2}(k + 2)$ or $\frac{1}{2}(k + 1)(k + 2)$ Factoring the right member of (3), we have

$$\frac{k}{2}(k+1)+(k+1)=(k+1)\left(\frac{k}{2}+1\right)$$
$$=(k+1)\left(\frac{k+2}{2}\right)$$
$$=\tfrac{1}{2}(k+1)(k+2)$$

The right members are identical

In Part I we verified the theorem for $n=1$ and $n=2$ If $k=2$, then $k+1=3$, and the theorem is valid for $n=3$ If $k=3$, then $k+1=4$, and the theorem is valid for $n=4$ In this manner we see that the theorem is always valid for the next consecutive integer, and therefore valid for all positive integers

Parts I and II of the proof are both necessary Part I alone is not sufficient For example, if we wish to prove that n^2-n+11 for any positive integral value of n is a prime number, we can verify the theorem for $n=1, 2, 3$, etc, up to 10, as shown in the following table

n	1	2	3	4	5	6	7	8	9	10
n^2-n+11	11	13	17	23	31	41	53	67	83	101

But when we try $n=11$, $n^2-n+11=121$, which is not a prime number

Part II alone is not sufficient proof For example, if we try to prove, n denoting any positive integer, that

$$2+4+6+\quad+2n=n(n+1)+4, \text{ we have}$$

if $n=k$ $\quad 2+4+6+\quad+2k=k(k+1)+4 \qquad (1)$

if $n=k+1$

$$2+4+6+\quad+2k+2(k+1)=(k+1)(k+2)+4 \qquad (2)$$

Adding $2(k+1)$ to both members of (1),

$$2+4+6+\quad+2k+2(k+1)=k(k+1)+4+2(k+1) \quad (3)$$

The left members of (2) and (3) are identical

Simplifying the right member of (2), we have

$$k^2+3k+2+4, \text{ or } k^2+3k+6$$

Simplifying the right member of (3), we have

$$k^2+k+4+2k+2, \text{ or } k^2+3k+6$$

Since the right members are identical, the theorem is valid for $n=k+1$ if it is valid for $n=k$ However, if we try to verify

the theorem for $n=1$, we have $2=1(1+1)+4$, or $2=6$, which is obviously untrue, and the theorem is false

Example 2. Prove for positive integral values of n that

$$1^2+2^2+3^2+\cdots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$

PROOF. *Part I.*

Let $n=1$ $\qquad 1^2=\tfrac{1}{6}(1+1)(2+1)$

$\qquad 1=1$

Let $n=2$ $\qquad 1^2+2^2=\tfrac{1}{6}(2)(2+1)(4+1)$

$\qquad 5=5$

Part II

Let $n=k$ $\quad 1^2+2^2+3^2+\cdots+k^2=\tfrac{1}{6}k(k+1)(2k+1)$ $\qquad (1)$

Let $n=k+1$ $\quad 1^2+2^2+3^2+\cdots+k^2+(k+1)^2$

$$=\tfrac{1}{6}(k+1)(k+2)(2k+3) \qquad (2)$$

Adding $(k+1)^2$ to both members of (1),

$$\begin{aligned}&1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\&=\tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2\end{aligned} \qquad (3)$$

The left members of (2) and (3) are identical

Transforming the right member of (3), we have

$$\begin{aligned}\tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2&=\tfrac{1}{6}(k+1)[k(2k+1)+6(k+1)]\\&=\tfrac{1}{6}(k+1)[2k^2+7k+6]\\&=\tfrac{1}{6}(k+1)(k+2)(2k+3)\end{aligned}$$

which is identical with the right member of (2).

Therefore the theorem is valid for all positive integral values of n

Example 3. *Prove* that the Binomial Theorem is valid for all integral values of n.

PROOF. *Part I.* The formula

$$(a+b)^n=a^n+\frac{na^{n-1}b}{1}+\frac{n(n-1)a^{n-2}b^2}{1\cdot 2}+\frac{n(n-1)(n-2)a^{n-3}b^3}{1\cdot 2\cdot 3}+\cdots$$

is valid for $n=6$. See page 393.

Part II. Suppose that the formula is valid for $n=k$ Then

$$(a+b)^k=a^k+\frac{ka^{k-1}b}{1}+\frac{k(k-1)a^{k-2}b^2}{1\cdot 2}+\frac{k(k-1)(k-2)a^{k-3}b^3}{1\cdot 2\cdot 3}+\cdots \qquad (1)$$

Multiply (1) by $a+b$,

$$(a+b)^{k+1}=a^{k+1}+ka^kb+\frac{k(k-1)a^{k-1}b^2}{1\cdot 2}+\frac{k(k-1)(k-2)a^{k-2}b^3}{1\cdot 2\cdot 3}+\cdots$$
$$+a^kb+ka^{k-1}b^2+\frac{k(k-1)a^{k-2}b^3}{1\cdot 2}+\cdots$$

Collecting like terms, we have

$$(a+b)^{k+1}=a^{k+1}+(k+1)a^kb+\frac{(k+1)ka^{k-1}b^2}{1\cdot 2}$$
$$+\frac{(k+1)(k)(k-1)a^{k-2}b^3}{1\cdot 2\cdot 3}+\cdots$$

Comparing this equation with that of the formula, we observe that they will be exactly alike if k in the formula is replaced by $k+1$. The theorem is therefore valid for the next higher power of n. It has been shown by multiplication that the theorem is valid for $n=6$. It is therefore valid for $n+1$ or 7, and if it is valid for 7, it is valid for $n+1$ or 8. Hence the theorem is valid for any integral power of n.

[A]

EXERCISES

Prove by mathematical induction for positive integral values of n.

1. $2+4+6+\cdots+2n=n(n+1)$
2. $3+6+9+\cdots+3n=\frac{3}{2}n(n+1)$
3. $1+4+7+\cdots+(3n-2)=\frac{1}{2}n(3n-1)$
4. $2^0+2^1+2^2+\cdots+2^{n-1}=2^n-1$
5. $2^1+2^2+2^3+\cdots+2^n=2^{n+1}-2$
6. $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$
7. $1+5+9+\cdots+(4n-3)=n(2n-1)$
8. $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+n(n+2)=\frac{1}{6}n(n+1)(2n+7)$
9. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

[B]

10. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$
11. $1^3+3^3+5^3+\cdots+(2n-1)^3=n^2(2n^2-1)$

12 $a + ar + ar^2 + \quad + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$

13 $a + (a + d) + (a + 2d) + \quad + [a + (n - 1)d]$
$= \dfrac{n}{2}[2a + (n - 1)d]$

Checking Your Understanding of Chapter 21

In this chapter you should have learned

1 That proof by mathematical induction consists in proving that the theorem or principle is true, for a particular case and then showing that if it is true for $n = k$, it is true for $n = k + 1$

2 How to prove some formulas by mathematical induction

CHAPTER TEST

[A]

Prove for positive integral values of n

1 $1 + 3 + 5 + \quad + (2n - 1) = n^2$

2 $3^1 + 3^2 + 3^3 + \quad + 3^n = \frac{3}{2}(3^n - 1)$

CHAPTER

22

Inequalities

In this chapter you study quantities which are not equal

If a and b are two unequal real numbers and $a - b$ is positive then a is greater than b $(a > b)$ If $a - b$ is negative then a is less than b $(a < b)$ Any relation involving only real numbers and expressed by the symbols $>$ or $<$ is an inequality

Sense of an Inequality[A]

Two inequalities have the *same sense* if their signs of inequality point in the same direction as $a > b$ and $c > d$ or $m < n$ and $x < y$ Two inequalities are *opposite in sense* if their signs of inequality point in opposite directions, as $a > b$ and $x < y$

Absolute and Conditional Inequalities[A]

An absolute inequality is one which is true for all real numbers Thus $3 < 5$ and $a^2 + 1 > 0$ are absolute inequalities A conditional inequality is one that is true only for certain values of the letter involved Thus $a^2 - 1 > 0$ is satisfied only by values of $a > 1$ or $a < -1$ It is not true for values of a from -1 to $+1$

Axioms of Inequality[A]

The following fundamental principles or theorems on inequalities are stated as axioms

1 The sense of an inequality is unchanged if the same number is added to (or subtracted from) both members of the inequality

Examples If $a > b$, then $a + c > b + c$
If $a > b$, then $a - c > b - c$

2 The sense of an inequality is unchanged if both members of the inequality are multiplied (or divided) by the same positive number

Example If $a > b$, and c is a positive number, then $ac > bc$, and $\frac{a}{c} > \frac{b}{c}$

3 The sense of an inequality is reversed if both members of the inequality are multiplied (or divided) by the same negative number

Example If $a > b$, and c is a negative number, then $ac < bc$, and $\frac{a}{c} < \frac{b}{c}$

4 The sense of an inequality whose members are positive quantities is unchanged if both members are raised to the same positive power (or the same positive root is extracted)

Examples If $2 < 3$, then $2^5 < 3^5$
If $25 > 16$, then $5 > 4$

Proof of Inequalities (8)

The best procedure in proving inequalities is to assume that the inequality is true and reduce it to a simpler inequality which can be verified Then reverse the steps, giving the correct reason for each step

Example 1 Prove that $a^2 + b^2 > 2ab$ if $a \neq b$

Analysis

$$\text{If } a^2 + b^2 > 2ab$$
$$a^2 - 2ab + b^2 > 0 \qquad \text{Axiom 1}$$
$$(a-b)^2 > 0$$

This last statement is true since $a \neq b$ and the square of any real number is positive

PROOF

$a \neq b$	Given
Then $(a-b)^2 > 0$	The square of any real number is greater than zero
$a^2 - 2ab + b^2 > 0$	
$a^2 + b^2 > 2ab$	Axiom 1

Example 2 If $a \neq b$, and a and b are positive quantities, prove that $\dfrac{2ab}{a+b} < \sqrt{ab}$

Analysis

$$\frac{4a^2b^2}{a^2+2ab+b^2} < ab \qquad \text{Axiom 4}$$
$$4a^2b^2 < a^3b + 2a^2b^2 + ab^3 \qquad \text{Axiom 2}$$
$$0 < a^3b - 2a^2b^2 + ab^3 \qquad \text{Axiom 1}$$
$$0 < ab(a^2 - 2ab + b^2)$$
$$0 < ab(a-b)^2$$

We know this last inequality to be true, since a and b are unequal and positive

PROOF a and b are positive and unequal

Then $0 < ab(a-b)^2$	The product of any three positive numbers is positive
$0 < a^3b - 2a^2b^2 + ab^3$	
$4a^2b^2 < a^3b + 2a^2b^2 + ab^3$	Axiom 1
$4a^2b^2 < ab(a^2 + 2ab + b^2)$	
$\dfrac{4a^2b^2}{(a+b)^2} < ab$	Axiom 2
$\dfrac{2ab}{a+b} < \sqrt{ab}$	Axiom 4

[B] EXERCISES

If a and b are unequal and positive real numbers prove the following

1 $\dfrac{a+b}{2} > \sqrt{ab}$

2 $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}$

3 $\sqrt{ab} > \dfrac{2ab}{a+b}$

4 $a^4 + b^4 > ab(a^2 + b^2)$

5 $a + \dfrac{1}{a} > 2$ if $a \neq 1$

6 $2(a^2 + b^2) > (a+b)^2$

7 $\dfrac{a^2}{b} + \dfrac{b^2}{a} > a + b$

8 $\dfrac{a^3 + b^3}{a^2 + b^2} > \dfrac{a^2 + b^2}{a+b}$

9 $a^3 - b^3 > (a-b)^3$ if $a > b$

10 $a^2(a+1) + b^2(b+1) > ab(a+b+2)$

Solution of Conditional Inequalities [A]

To solve a conditional inequality involving one variable means to find all the real values of the variable for which the inequality is true. We shall consider only inequalities in one variable. Every inequality can be put in the form $f(x) > 0$ or $f(x) < 0$

Conditional inequalities may be solved algebraically or graphically. Two examples will be solved by both methods

Example 1. Solve $3x - 5 > x + 2$

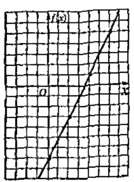
Algebraic Solution

$$3x - 5 > x + 2$$
$$3x - x > 5 + 2$$
$$2x > 7$$
$$x > 3\tfrac{1}{2}$$

Graphical Solution

$$3x - 5 > x + 2$$
$$3x - 5 - x - 2 > 0$$
$$2x - 7 > 0$$

Draw the graph for $f(x) = 2x - 7$



x	2	4	6
f(x)	-3	1	5

The inequality $3x - 5 > x + 2$ is true for all values of x for which the graph of $f(x) = 2x - 7$ lies above the x-axis, that is, it is true for $x > 3\frac{1}{2}$

Example 2 Solve $x^2 - x > 6$

Algebraic Solution

$$x^2 - x > 6$$
$$x^2 - x - 6 > 0$$

Factoring the left member, $(x - 3)(x + 2) > 0$

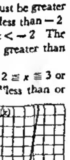
In order that the inequality may be true, either both factors must be positive or both factors must be negative, so that their product is positive For both factors to be positive x must be greater than 3 For both factors to be negative x must be less than -2 Therefore the inequality is true when $x > 3$ and $x < -2$ The inequality is not true for the range of values of x greater than or equal to -2 and less than or equal to 3

This last statement is written symbolically either $-2 \leqq x \leqq 3$ or $-2 \leq x \leq 3$ The symbols $\leqq$ and $\leq$ are read "less than or equal to"

Graphical Solution.

$$x^2 - x > 6$$
$$x^2 - x - 6 > 0$$

Let $f(x) = x^2 - x - 6$



x	-3	-2	0	3	4
f(x)	6	0	-6	0	6

The inequality $x^2 - x > 6$ is true for all values of x for which the graph of $f(x) = x^2 - x - 6$ lies above the x axis, that is, when $x > 3$ and $x < -2$. The inequality is not true for the values of x for which the graph of $f(x) = x^2 - x - 6$ lies below the x axis, that is, for $-2 \leqq x \leqq 3$

Usually the graphical solution is much more satisfactory than an algebraic solution

[A] **EXERCISES**

Solve algebraically

1. $3x > 12$
2. $2x < -6$
3. $6x - 3 < x + 2$
4. $\frac{2x-3}{4} > \frac{x-1}{3}$
5. $\frac{3x+1}{5} < \frac{2x-3}{3}$
6. $\frac{x}{3} + 3x > 2x + \frac{4}{5}$
7. $x(x-2) + 5 > x(x-3)$
8. $(x+6)(x-1) < 2x^2 - x(x+1)$
9. $(x-1)(x+3) > 0$
10. $x^2 - 4 > 0$
11. $x^2 - x - 12 > 0$

Solve graphically

12. $2x - 7 > x - 3$
13. $x^2 + 8x > 9$
14. $(x-2)(x+5) > 0$
15. $2x(x+3) > x(x-1) - 8$
16. $2x^2 - 4x + 3 > 0$
17. $\frac{1}{x^2} < \frac{1}{4}$
18. $4 + x > x^2$
19. $x + 3 > \frac{4}{x}$
20. $x(x-6) > 6$

Checking Your Understanding of Chapter 22

Before you leave this chapter make sure that you

1. Know what an inequality is, and can distinguish between absolute and conditional inequalities (p 555)
2. Understand the axioms of inequality, and can use them (pp 555–556)
3. Can prove simple inequalities such as those given in this chapter (p 556)
4. Can solve conditional inequalities both algebraically and geometrically (p. 557)

CHAPTER TESTS

[Test 1]

1 Tell whether the following inequalities are absolute or conditional

a $7 > 4$ b $x + 3 > 2$ c $6 > x^2 + 2$ d $a^2 > 0$

2 Is $3 + 2i > 5 - i$ an inequality?

3 If $-3x > 6$, is $x < -2$?

4 Solve $(x + 2)^2 > x^2 + 6$

5 Solve graphically $x^2 - 2x > 10$

[Test 2]

1 Solve graphically $7x + 3 > 10$

2 Solve graphically $x^2 + 4x > 2$

3 Solve graphically $6x^2 - 5x > 6$

4 If $a > 0$ and $b > 0$, show that $\frac{a}{b} + \frac{b}{a} > -2$

5 *If $a > 0$, $b > 0$, and $a \neq b$, show that $a^2 + b^2 + 1 > ab + a + b$*

CUMULATIVE TESTS (CHAPTERS 15–22)

[B]

Solve each problem and write the answer (a), (b), (c), or (d) which you think is correct Give only one answer for each problem

1 The roots of the equation $x^2 - 5x + 3 = 0$ are

a real, rational, and equal

b imaginary

c real, rational, and unequal

d real, irrational, and unequal

2. For the roots of $4x^2 - 6x + k = 0$ to be equal, k equals

a. 2 c 0

b. $2\frac{1}{4}$ d Not (a), (b), or (c)

3 The equation whose roots are $\sqrt{3}$ and $-\sqrt{3}$ is

a. $x^2 - 3 = 0$ c. $x^2 + 2x - 3 = 0$

b $x^2 - \sqrt{3}x + 3 = 0$ d $x^2 + 3 = 0$

4. The product of the roots of $(a-b)x^2+bx+a^2-b^2=0$ is

a. $\dfrac{b}{a-b}$ c. $a-b$

b. $\dfrac{b}{b-a}$ d. $a+b$

5. The value of the determinant $\begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 5 \\ 1 & 2 & 1 \end{vmatrix}$ is

a. 6 c. 3

b. -1 d Not (a), (b), or (c)

6. Transform the determinant

$$\begin{vmatrix} x & y & 2 \\ 1 & 2 & 2 \\ -2 & 1 & 3 \end{vmatrix} = 0$$

to show (without expansion) that a solution is $x=1, y=2$

a. $\begin{vmatrix} x-y & y & 2 \\ -1 & 2 & 2 \\ -3 & 1 & 3 \end{vmatrix}$ c. $\begin{vmatrix} x & y-2 & 2 \\ 1 & 0 & 2 \\ -2 & -2 & 3 \end{vmatrix}$

b. $\begin{vmatrix} x-1 & y-2 & 0 \\ 1 & 2 & 2 \\ -2 & 1 & 3 \end{vmatrix}$ d. Not (a), (b), or (c)

7. *In how many ways can 6 front seats of a classroom be* filled by selections from 8 students?

a. 18960 b. 20160 c. 2370 d. 8

8. From a suit of 13 playing cards, how many different hands of 6 cards each can be dealt to a player?

a. 1716 c. 78

b. 1287 d. Not (a), (b), or (c)

9. If three coins are tossed into the air simultaneously, what is the chance that two, and not more than two, will turn up heads?

a. $\frac{1}{8}$ b. $\frac{3}{8}$ c. $\frac{2}{3}$ d. $\frac{1}{2}$

Problems **10**, **11** and **12** refer to the table below, which gives the number of geometry test papers with grades of 70% 80%, 85% 90% and 100%

Grades	Number
70%	7
80%	8
85%	6
90%	10
100%	2

10 The mode is

a 75% c 85%
b 80% d 90%

11 The median grade is

a 80% c 85%
b $83\frac{1}{3}$% d Not (a), (b), or (c)

12 The arithmetic mean grade is

a 83% c 85%
b $83\frac{1}{3}$% d $83\frac{1}{33}$%

13 Find the y-intercept of the tangent to the graph of $y = \frac{1}{2}x^2 - 6x$ at the point $(8, -16)$

a 8 c -16
b -32 d -8

14 Find the maximum value of y if $y = 9x - 6x^2$

a $\frac{3}{2}$ c $-\frac{3}{4}$
b $-\frac{2}{3}$ d $3\frac{3}{8}$

15 Transform the equation $x^3 - 4x^2 - x + 12 = 0$ into a new equation whose roots are 2 less than those of the original equation

a $x^3 - 2x^2 - 4x = 0$ c $x^3 + 2x^2 - 5x + 2 = 0$
b $x^3 + 2x^2 = 0$ d Not (a), (b), or (c)

16. Find the roots of the equation $2x^3 - x^2 + x - 2 = 0$

a. $1, -2, 1$ b. $2, 3, 0$

c. $1, \frac{-1 \pm 15i}{2}$ d. $1, \frac{-1 \pm 15i}{4}$

17. One root of $x^3 + x^2 - x + 2 = 0$ is $\frac{1 + i\sqrt{3}}{2}$. Find the two other roots.

a. $\frac{1 - i\sqrt{3}}{2}, -2$ c. $1, 2$

b. $\frac{1 + i\sqrt{3}}{2}, +2$ d. $1, \frac{-1 \pm i\sqrt{15}}{4}$

18. When the roots of $x^4 - x^2 - 7x = 0$ are diminished by 3,

a. the x-axis is raised 3 units.

b. the y-axis is moved to the right 3 units

c. the y-axis is moved to the left 3 units

d. Not (a), (b), or (c)

19. If $a \neq b$ and a and b are positive quantities, then

a. $\frac{2ab}{a+b} > \frac{a+b}{2}$ c. $\frac{a+b}{2} < \sqrt{ab}$

b. $\frac{2ab}{a+b} > \sqrt{ab}$ d. $\frac{2ab}{a+b} < \sqrt{ab}$

20. Solve $x^2 - x > 20$.

a. $x > 5$ and $x < -4$ c. $x > 4$ and $x < -5$

b. $x < 5$ and $x > -4$ d. $x > 4$ and $x < -4$

Table 1 Square Roots and Cube Roots

N	$\sqrt{N}$	$\sqrt[3]{N}$	N	$\sqrt{N}$	$\sqrt[3]{N}$	N	$\sqrt{N}$	$\sqrt[3]{N}$	N	$\sqrt{N}$	$\sqrt[3]{N}$
1	1.000	1.000	51	7.141	3.708	101	10.050	4.657	151	12.288	5.325
2	1.414	1.260	52	7.211	3.733	102	10.100	4.672	152	12.329	5.337
3	1.732	1.442	53	7.280	3.756	103	10.149	4.688	153	12.369	5.348
4	2.000	1.587	54	7.348	3.780	104	10.198	4.703	154	12.410	5.360
5	2.236	1.710	55	7.416	3.803	105	10.247	4.718	155	12.450	5.372
6	2.449	1.817	56	7.483	3.826	106	10.296	4.733	156	12.490	5.383
7	2.646	1.913	57	7.550	3.849	107	10.344	4.747	157	12.530	5.395
8	2.828	2.000	58	7.616	3.871	108	10.392	4.762	158	12.570	5.406
9	3.000	2.080	59	7.681	3.893	109	10.440	4.777	159	12.610	5.418
10	3.162	2.154	60	7.746	3.915	110	10.488	4.791	160	12.649	5.429
11	3.317	2.224	61	7.810	3.936	111	10.536	4.806	161	12.689	5.440
12	3.464	2.289	62	7.874	3.958	112	10.583	4.820	162	12.728	5.451
13	3.606	2.351	63	7.937	3.979	113	10.630	4.835	163	12.767	5.463
14	3.742	2.410	64	8.000	4.000	114	10.677	4.849	164	12.806	5.474
15	3.873	2.466	65	8.062	4.021	115	10.724	4.863	165	12.845	5.485
16	4.000	2.520	66	8.124	4.041	116	10.770	4.877	166	12.884	5.496
17	4.123	2.571	67	8.185	4.062	117	10.817	4.891	167	12.923	5.507
18	4.243	2.621	68	8.246	4.082	118	10.863	4.905	168	12.961	5.518
19	4.359	2.668	69	8.307	4.102	119	10.909	4.919	169	13.000	5.529
20	4.472	2.714	70	8.367	4.121	120	10.955	4.932	170	13.038	5.540
21	4.583	2.759	71	8.426	4.141	121	11.000	4.946	171	13.077	5.550
22	4.690	2.802	72	8.485	4.160	122	11.045	4.960	172	13.115	5.561
23	4.796	2.844	73	8.544	4.179	123	11.091	4.973	173	13.153	5.572
24	4.899	2.884	74	8.602	4.198	124	11.136	4.987	174	13.191	5.583
25	5.000	2.925	75	8.660	4.217	125	11.180	5.000	175	13.229	5.593
26	5.099	2.962	76	8.718	4.236	126	11.225	5.013	176	13.266	5.604
27	5.196	3.000	77	8.775	4.254	127	11.269	5.027	177	13.304	5.615
28	5.292	3.037	78	8.832	4.273	128	11.314	5.040	178	13.342	5.625
29	5.385	3.072	79	8.888	4.291	129	11.358	5.053	179	13.379	5.636
30	5.477	3.107	80	8.944	4.309	130	11.402	5.066	180	13.416	5.646
31	5.568	3.141	81	9.000	4.327	131	11.446	5.079	181	13.454	5.657
32	5.657	3.175	82	9.055	4.344	132	11.489	5.092	182	13.491	5.667
33	5.745	3.208	83	9.110	4.362	133	11.533	5.104	183	13.528	5.677
34	5.831	3.240	84	9.165	4.380	134	11.576	5.117	184	13.565	5.688
35	5.916	3.271	85	9.220	4.397	135	11.619	5.130	185	13.601	5.698
36	6.000	3.302	86	9.274	4.414	136	11.662	5.143	186	13.638	5.708
37	6.083	3.332	87	9.327	4.431	137	11.705	5.155	187	13.675	5.718
38	6.164	3.362	88	9.381	4.448	138	11.747	5.168	188	13.711	5.729
39	6.245	3.391	89	9.434	4.465	139	11.790	5.180	189	13.748	5.739
40	6.325	3.420	90	9.487	4.481	140	11.832	5.192	190	13.784	5.748
41	6.403	3.448	91	9.539	4.498	141	11.874	5.205	191	13.820	5.759
42	6.481	3.476	92	9.592	4.514	142	11.916	5.217	192	13.856	5.769
43	6.557	3.503	93	9.644	4.531	143	11.958	5.229	193	13.892	5.779
44	6.633	3.530	94	9.695	4.547	144	12.000	5.241	194	13.928	5.789
45	6.708	3.557	95	9.747	4.563	145	12.042	5.254	195	13.964	5.799
46	6.782	3.583	96	9.798	4.579	146	12.083	5.266	196	14.000	5.809
47	6.856	3.609	97	9.849	4.595	147	12.124	5.278	197	14.036	5.819
48	6.928	3.634	98	9.899	4.610	148	12.166	5.290	198	14.071	5.828
49	7.000	3.658	99	9.950	4.626	149	12.207	5.301	199	14.107	5.838
50	7.071	3.684	100	10.000	4.642	150	12.247	5.313	200	14.142	5.848

Table II Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	83 6	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table III Values of Trigonometric Functions

Angle	sin	tan		cos	Angle
0° 0′	0.0000	0.0000	—	1.0000	90° 0′
10	0.0029	0.0029	343.77	1.0000	30
20	0.0058	0.0058	171.89	1.0000	40
30	0.0087	0.0087	114.59	1.0000	30
40	0.0116	0.0116	85.940	0.9999	20
50	0.0145	0.0145	68.750	0.9999	10
1° 0	0.0175	0.0175	57.290	0.9998	89° 0
10	0.0204	0.0204	49.104	0.9998	50
20	0.0233	0.0233	42.964	0.9997	40
30	0.0262	0.0262	38.188	0.9997	30
40	0.0291	0.0291	34.368	0.9996	20
50	0.0320	0.0320	31.242	0.9995	10
2° 0′	0.0349	0.0349	28.636	0.9994	88° 0
10	0.0378	0.0378	26.432	0.9993	50
20	0.0407	0.0407	24.542	0.9992	40
30	0.0436	0.0437	22.904	0.9990	30
40	0.0465	0.0466	21.470	0.9989	20
50	0.0494	0.0495	20.206	0.9988	10
3° 0′	0.0523	0.0524	19.081	0.9986	87° 0
10	0.0552	0.0553	18.075	0.9985	50
20	0.0581	0.0582	17.169	0.9983	40
30	0.0610	0.0612	16.350	0.9981	30
40	0.0640	0.0641	15.605	0.9980	20
50	0.0669	0.0670	14.924	0.9978	10
4° 0′	0.0698	0.0699	14.301	0.9976	86° 0
10	0.0727	0.0729	13.727	0.9974	50
20	0.0756	0.0758	13.197	0.9971	40
30	0.0785	0.0787	12.706	0.9969	30
40	0.0814	0.0816	12.251	0.9967	20
50	0.0843	0.0846	11.826	0.9964	10
5° 0′	0.0872	0.0875	11.430	0.9962	85° 0′
10	0.0901	0.0904	11.059	0.9959	50
20	0.0929	0.0934	10.712	0.9957	40
30	0.0958	0.0963	10.385	0.9954	30
40	0.0987	0.0992	10.078	0.9951	20
50	0.1016	0.1022	9.7882	0.9948	10
6° 0′	0.1045	0.1051	9.5144	0.9945	84° 0′
10	0.1074	0.1080	9.2553	0.9942	50
20	0.1103	0.1110	9.0098	0.9939	40
30	0.1132	0.1139	8.7769	0.9936	30
40	0.1161	0.1169	8.5555	0.9932	20
50	0.1190	0.1198	8.3450	0.9929	10
7° 0′	0.1219	0.1228	8.1443	0.9925	83° 0
10	0.1248	0.1257	7.9530	0.9922	50
20	0.1276	0.1287	7.7704	0.9918	40
50	0.1305	0.1317	7.5958	0.9914	30
Angle	cos		tan	sin	Angle

Angle	sin	tan		cos	Angle
7° 30	0.1305	0.1317	7.5958	0.9914	82° 30′
40	0.1334	0.1346	7.4287	0.9911	20
50	0.1363	0.1376	7.2687	0.9907	10
8° 0′	0.1392	0.1405	7.1154	0.9903	82° 0′
10	0.1421	0.1435	6.9682	0.9899	50
20	0.1449	0.1465	6.8269	0.9894	40
30	0.1478	0.1495	6.6912	0.9890	30
40	0.1507	0.1524	6.5606	0.9886	20
50	0.1536	0.1554	6.4348	0.9881	10
9° 0′	0.1564	0.1584	6.3138	0.9877	81° 0′
10	0.1593	0.1614	6.1970	0.9872	50
20	0.1622	0.1644	6.0844	0.9868	40
30	0.1650	0.1673	5.9758	0.9863	30
40	0.1679	0.1703	5.8708	0.9858	20
50	0.1708	0.1733	5.7694	0.9853	10
10° 0′	0.1736	0.1763	5.6713	0.9848	80° 0′
10	0.1765	0.1793	5.5764	0.9843	50
20	0.1794	0.1823	5.4845	0.9838	40
30	0.1822	0.1853	5.3955	0.9833	30
40	0.1851	0.1883	5.3093	0.9827	20
50	0.1880	0.1914	5.2257	0.9822	10
11° 0	0.1908	0.1944	5.1446	0.9816	79° 0′
10	0.1937	0.1974	5.0658	0.9811	50
20	0.1965	0.2004	4.9894	0.9805	40
30	0.1994	0.2035	4.9152	0.9799	30
40	0.2022	0.2065	4.8430	0.9793	20
50	0.2051	0.2095	4.7729	0.9787	10
12° 0	0.2079	0.2126	4.7046	0.9781	78° 0′
10	0.2108	0.2156	4.6382	0.9775	50
20	0.2136	0.2186	4.5736	0.9769	40
30	0.2164	0.2217	4.5107	0.9763	30
40	0.2193	0.2247	4.4494	0.9757	20
50	0.2221	0.2278	4.3897	0.9750	10
13° 0′	0.2250	0.2309	4.3315	0.9744	77° 0
10	0.2278	0.2339	4.2747	0.9737	50
20	0.2306	0.2370	4.2193	0.9730	40
30	0.2334	0.2401	4.1653	0.9724	30
40	0.2363	0.2432	4.1126	0.9717	20
50	0.2391	0.2462	4.0611	0.9710	10
14° 0	0.2419	0.2493	4.0108	0.9703	76° 0
10	0.2447	0.2524	3.9617	0.9696	50
20	0.2476	0.2555	3.9136	0.9689	40
30	0.2504	0.2586	3.8667	0.9681	30
40	0.2532	0.2617	3.8208	0.9674	20
50	0.2560	0.2648	3.7760	0.9667	10
15° 0′	0.2588	0.2679	3.7321	0.9659	75° 0′
Angle	cos		tan	sin	Angle

Angle	sin	tan		cos	Angle
15° 0'	0.2588	0.2679	3.7321	0.9659	75° 0'
10	0.2616	0.2711	3.6891	0.9652	50
20	0.2644	0.2742	3.6470	0.9644	40
30	0.2672	0.2773	3.6059	0.9636	30
40	0.2700	0.2805	3.5656	0.9628	20
50	0.2728	0.2836	3.5261	0.9621	10
16° 0'	0.2756	0.2867	3.4874	0.9613	74° 0'
10	0.2784	0.2899	3.4495	0.9605	50
20	0.2812	0.2931	3.4124	0.9596	40
30	0.2840	0.2962	3.3759	0.9588	30
40	0.2868	0.2994	3.3402	0.9580	20
50	0.2896	0.3026	3.3052	0.9572	10
17° 0'	0.2924	0.3057	3.2709	0.9563	73° 0'
10	0.2952	0.3089	3.2371	0.9555	50
20	0.2979	0.3121	3.2041	0.9546	40
30	0.3007	0.3153	3.1716	0.9537	30
40	0.3035	0.3185	3.1397	0.9528	20
50	0.3062	0.3217	3.1084	0.9520	10
18° 0	0.3090	0.3249	3.0777	0.9511	72° 0
10	0.3118	0.3281	3.0475	0.9502	50
20	0.3145	0.3314	3.0178	0.9492	40
30	0.3173	0.3346	2.9887	0.9483	30
40	0.3201	0.3378	2.9600	0.9474	20
50	0.3228	0.3411	2.9319	0.9465	10
19° 0'	0.3256	0.3443	2.9042	0.9455	71° 0
10	0.3283	0.3476	2.8770	0.9446	50
20	0.3311	0.3508	2.8502	0.9436	40
30	0.3338	0.3541	2.8239	0.9426	30
40	0.3365	0.3574	2.7980	0.9417	20
50	0.3393	0.3607	2.7725	0.9407	10
20° 0'	0.3420	0.3640	2.7475	0.9397	70° 0
10	0.3448	0.3673	2.7228	0.9387	50
20	0.3475	0.3706	2.6985	0.9377	40
30	0.3502	0.3739	2.6746	0.9367	30
40	0.3529	0.3772	2.6511	0.9356	20
50	0.3557	0.3805	2.6279	0.9346	10
21° 0'	0.3584	0.3839	2.6051	0.9336	69° 0
10	0.3611	0.3872	2.5826	0.9325	50
20	0.3638	0.3906	2.5605	0.9315	40
30	0.3665	0.3939	2.5386	0.9304	30
40	0.3692	0.3973	2.5172	0.9293	20
50	0.3719	0.4006	2.4960	0.9283	10
22° 0'	0.3746	0.4040	2.4751	0.9272	68° 0'
10	0.3773	0.4074	2.4545	0.9261	50
20	0.3800	0.4108	2.4342	0.9250	40
30	0.3827	0.4142	2.4142	0.9239	30
Angle	cos		tan	sin	Angle

Angle	sin	tan		cos	Angle
22° 30'	0.3827	0.4142	2.4142	0.9239	67° 30'
40	0.3854	0.4176	2.3945	0.9228	20
50	0.3881	0.4210	2.3750	0.9216	10
23° 0'	0.3907	0.4245	2.3559	0.9205	67° 0'
10	0.3934	0.4279	2.3369	0.9194	50
20	0.3961	0.4314	2.3183	0.9182	40
30	0.3987	0.4348	2.2998	0.9171	30
40	0.4014	0.4383	2.2817	0.9159	20
50	0.4041	0.4417	2.2637	0.9147	10
24° 0'	0.4067	0.4452	2.2460	0.9135	66° 0
10	0.4094	0.4487	2.2286	0.9124	50
20	0.4120	0.4522	2.2113	0.9112	40
30	0.4147	0.4557	2.1943	0.9100	30
40	0.4173	0.4592	2.1775	0.9088	20
50	0.4200	0.4628	2.1609	0.9075	10
25° 0'	0.4226	0.4663	2.1445	0.9063	65° 0'
10	0.4253	0.4699	2.1283	0.9051	50
20	0.4279	0.4734	2.1123	0.9038	40
30	0.4305	0.4770	2.0965	0.9026	30
40	0.4331	0.4806	2.0809	0.9013	20
50	0.4358	0.4841	2.0655	0.9001	10
26° 0'	0.4384	0.4877	2.0503	0.8988	64° 0'
10	0.4410	0.4913	2.0353	0.8975	50
20	0.4436	0.4950	2.0204	0.8962	40
30	0.4462	0.4986	2.0057	0.8949	30
40	0.4488	0.5022	1.9912	0.8936	20
50	0.4514	0.5059	1.9768	0.8923	10
27° 0'	0.4540	0.5095	1.9626	0.8910	63° 0'
10	0.4566	0.5132	1.9486	0.8897	50
20	0.4592	0.5169	1.9347	0.8884	40
30	0.4617	0.5206	1.9210	0.8870	30
40	0.4643	0.5243	1.9074	0.8857	20
50	0.4669	0.5280	1.8940	0.8843	10
28° 0'	0.4695	0.5317	1.8807	0.8829	62° 0
10	0.4720	0.5354	1.8676	0.8816	50
20	0.4746	0.5392	1.8546	0.8802	40
30	0.4772	0.5430	1.8418	0.8788	30
40	0.4797	0.5467	1.8291	0.8774	20
50	0.4823	0.5505	1.8165	0.8760	10
29° 0'	0.4848	0.5543	1.8040	0.8746	61° 0'
10	0.4874	0.5581	1.7917	0.8732	50
20	0.4899	0.5619	1.7796	0.8718	40
30	0.4924	0.5658	1.7675	0.8704	30
40	0.4950	0.5696	1.7556	0.8689	20
50	0.4975	0.5735	1.7437	0.8675	10
30° 0'	0.5000	0.5774	1.7321	0.8660	60° 0'
Angle	cos		tan	sin	Angle

Values of Trigonometric Functions

ANGLE	SIN	TAN		COS	ANGLE
30° 0′	0 5000	0 5774	1 7321	0 8660	60° 0′
10	0 5025	0 6012	1 7205	0 8646	60
20	0 5050	0 6051	1 7090	0 6631	40
30	0 5075	0 5890	1 6977	0 8816	30
40	0 5100	0 5930	1 6864	0 5601	20
60	0 5125	0 5969	1 6753	0 5587	10
31° 0′	0 5150	0 6009	1 6643	0 8572	59° 0′
10	0 5175	0 6045	1 6534	0 6557	60
20	0 5200	0 6058	1 6426	0.8542	40
30	0.5225	0 6125	1 6319	0 8526	30
40	0 5250	0 6168	1 6212	0 8511	20
60	0 5275	0 6205	1 6107	0 5496	10
32° 0	0 5299	0 6249	1 6003	0 5460	58° 0
10	0 5324	0 6269	1 5900	0 8465	60
20	0 5348	0 6330	1 5798	0.8450	40
30	0 8373	0 6371	1 5697	0 6434	50
40	0 5398	0 6412	1 5597	0 8416	20
60	0 5422	0 6453	1 5497	0 8403	10
33° 0	0 5446	0 6494	1 5399	0 8387	57° 0
10	0 5471	0 6536	1 6301	0.8371	60
20	0 5495	0 6577	1 5204	0 8355	40
30	0 5519	0 6619	1 5106	0.8339	30
40	0 5544	0 6861	1 5013	0.8323	20
60	0 5560	0 6703	1 4919	0 8307	10
34° 0	0 5592	0 6745	1 4826	0 5290	56° 0
10	0 5616	0 8767	1.4733	0 5274	60
20	0 5640	0 6830	1 4641	0 8258	40
50	0 5664	0 6873	1 4550	0 5241	30
40	0 5686	0 6916	1 4460	0 8225	20
50	0 6712	0 6959	1 4370	0.8208	10
35° 0	0 5736	0 7002	1 4281	0 8192	55° 0
10	0 8760	0 7046	1 4193	0.8175	60
20	0 5783	0 7089	1 4106	0.8158	40
30	0 5807	0 7133	1 4016	0 6141	50
40	0 5831	0 7177	1.3934	0 8124	20
60	0 5864	0 7221	1.3848	0.8107	30
36° 0	0 6078	0 7265	1 6764	0 8090	54° 0
10	0 5901	0 7810	1.8680	0.8073	60
20	0 5925	0 7355	1.8597	0.8056	40
60	0 5945	0 7400	1.3614	0 5036	80
40	0.5972	0 7445	1.3432	0.8021	20
50	0 5995	0 7490	1.8351	0 5004	10
37° 0′	0 6018	0 7536	1.3270	0 7985	53° 0
10	0 6041	0 7601	1.3190	0 7969	50
20	0 6060	0 7627	1.3111	0 7951	40
80	0 6093	0 7873	1.8032	0 7934	50
ANGLE	COS		TAN	SIN	ANGLE

ANGLE	SIN	TAN		COS	ANGLE
37° 30	0 6088	0 7673	1 3032	0 7934	52° 30
40	0 6111	0 7720	1.2954	0 7916	20
60	0 6134	0 7766	1.2876	0 7598	10
38° 0	0 6157	0 7813	1 2799	0 7880	52° 0
10	0 6180	0 7860	1.2723	0 7562	60
20	0 6202	0 7907	1.2647	0 7844	40
20	0 6225	0 7654	1 2572	0 7626	30
40	0 6248	0 8002	1 2497	0 7808	20
60	0 6271	0 6050	1 2423	0 7790	10
39° 0	0 6293	0 8098	1 2349	0 7771	51° 0
10	0 6316	0 8146	1 2276	0 7753	60
20	0 6336	0 8195	1.2203	0 7735	40
30	0 6361	0 8243	1 2131	0 7716	30
40	0 6383	0 8292	1.2059	0 7895	20
60	0 6406	0.8342	1 1966	0 7679	10
40° 0	0 6428	0 8391	1 1915	0 7660	50° 0
10	0 6450	0 8441	1 1847	0 7642	60
20	0 6472	0 8491	1.1778	0 7623	40
60	0 6494	0 8541	1 1708	0 7604	30
40	0 6517	0 8591	1 1640	0 7586	20
60	0 6539	0 6642	1 1571	0 7566	10
41° 0	0 8561	0 8693	1 1504	0 7547	49° 0
10	0 6583	0 8744	1 1436	0 7528	60
20	0 6004	0.8796	1 1369	0.7609	40
30	0 8626	0 8847	1 1303	0 7490	30
40	0 6648	0 8899	1 1237	0 7470	20
60	0 6670	0.8952	1 1171	0 7451	10
42° 0	0 6691	0 9004	1 1106	0 7431	48° 0
10	0 6712	0 9057	1 1041	0 7412	60
20	0 6734	0 9110	1 0977	0 7392	40
30	0 6756	0 9163	1 0913	0 7373	50
40	0 6777	0 9217	1.0850	0.7353	20
30	0 6786	0 9271	1.0786	0 7333	10
43° 0′	0 6820	0 9325	1.0724	0 7314	47° 0
10	0 6841	0 9380	1.0661	0 7294	60
20	0.6862	0 9435	1 0595	0 7274	40
30	0 6864	0 6490	1 0538	0 7254	60
40	0 6905	0 9545	1.0477	0 7234	20
60	0 6926	0 6601	1.0416	0 7214	10
44° 0′	0 6947	0 6657	1 0355	0 7193	46° 0
10	0 6967	0 6713	1.0295	0 7173	50
20	0 5968	0.9770	1.0235	0 71[illegible]	40
60	0 7009	0 9827	1 0176	0 7133	60
40	0 7030	0 9884	1 0117	0 7112	20
60	0.7050	0.9942	1.0058	0.7092	10
45° 0′	0.7071	1.0000	1 0000	0 7071	45° 0′
ANGLE	COS		TAN	SIN	ANGLE

Table IV · Logarithms of Trigonometric Functions

Angle	Log sin	Log tan		Log cos	Angle
0° 0'	—	—	—	10.0000	90° 0'
10	7.4637	7.4637	2.5363	10.0000	50
20	7.7648	7.7648	2.2352	10.0000	40
30	7.9408	7.9409	2.0591	10.0000	30
40	8.0658	8.0658	1.9342	10.0000	20
50	8.1627	8.1627	1.8373	10.0000	10
1° 0'	8.2419	8.2419	1.7581	9.9999	89° 0'
10	8.3088	8.3089	1.6911	9.9999	50
20	8.3668	8.3669	1.6331	9.9999	40
30	8.4179	8.4181	1.5819	9.9999	30
40	8.4637	8.4638	1.5362	9.9998	20
50	8.5050	8.5053	1.4947	9.9998	10
2° 0'	8.5428	8.5431	1.4569	9.9997	88° 0'
10	8.5776	8.5779	1.4221	9.9997	50
20	8.6097	8.6101	1.3899	9.9996	40
30	8.6397	8.6401	1.3599	9.9996	30
40	8.6677	8.6682	1.3318	9.9995	20
50	8.6940	8.6945	1.3055	9.9995	10
3° 0'	8.7188	8.7194	1.2806	9.9994	87° 0'
10	8.7423	8.7429	1.2571	9.9993	50
20	8.7645	8.7652	1.2348	9.9993	40
30	8.7857	8.7865	1.2135	9.9992	30
40	8.8059	8.8067	1.1933	9.9991	20
50	8.8251	8.8261	1.1739	9.9990	10
4° 0'	8.8436	8.8446	1.1554	9.9989	86° 0'
10	8.8613	8.8624	1.1376	9.9989	50
20	8.8783	8.8795	1.1205	9.9988	40
30	8.8946	8.8960	1.1040	9.9987	30
40	8.9104	8.9118	1.0882	9.9986	20
50	8.9256	8.9272	1.0728	9.9985	10
5° 0'	8.9403	8.9420	1.0580	9.9983	85° 0'
10	8.9545	8.9563	1.0437	9.9982	50
20	8.9682	8.9701	1.0299	9.9981	40
30	8.9816	8.9836	1.0164	9.9980	30
40	8.9945	8.9966	1.0034	9.9979	20
50	9.0070	9.0093	0.9907	9.9977	10
6° 0'	9.0192	9.0216	0.9784	9.9976	84° 0'
10	9.0311	9.0336	0.9664	9.9975	50
20	9.0426	9.0453	0.9547	9.9973	40
30	9.0539	9.0567	0.9433	9.9972	30
40	9.0648	9.0678	0.9322	9.9971	20
50	9.0755	9.0786	0.9214	9.9969	10
7° 0'	9.0859	9.0891	0.9109	9.9968	83° 0'
10	9.0961	9.0995	0.9005	9.9966	50
20	9.1060	9.1096	0.8904	9.9964	40
30	9.1157	9.1194	0.8806	9.9963	30
Angle	Log cos		Log tan	Log sin	Angle

Angle	Log sin	Log tan		Log cos	Angle
7° 30'	9.1157	9.1194	0.8806	9.9963	82° 30'
40	9.1252	9.1291	0.8709	9.9961	20
50	9.1345	9.1385	0.8615	9.9959	10
8° 0'	9.1436	9.1478	0.8522	9.9958	82° 0'
10	9.1525	9.1569	0.8431	9.9956	50
20	9.1612	9.1658	0.8342	9.9954	40
30	9.1697	9.1745	0.8255	9.9952	30
40	9.1781	9.1831	0.8169	9.9950	20
50	9.1863	9.1915	0.8085	9.9948	10
9° 0'	9.1943	9.1997	0.8003	9.9946	81° 0'
10	9.2022	9.2078	0.7922	9.9944	50
20	9.2100	9.2158	0.7842	9.9942	40
30	9.2176	9.2236	0.7764	9.9940	30
40	9.2251	9.2313	0.7687	9.9938	20
50	9.2324	9.2389	0.7611	9.9936	10
10° 0'	9.2397	9.2463	0.7537	9.9934	80° 0'
10	9.2468	9.2536	0.7464	9.9931	50
20	9.2538	9.2609	0.7391	9.9929	40
30	9.2606	9.2680	0.7320	9.9927	30
40	9.2674	9.2750	0.7250	9.9924	20
50	9.2740	9.2819	0.7181	9.9922	10
11° 0'	9.2806	9.2887	0.7113	9.9919	79° 0'
10	9.2870	9.2953	0.7047	9.9917	50
20	9.2934	9.3020	0.6980	9.9914	40
30	9.2997	9.3085	0.6915	9.9912	30
40	9.3058	9.3149	0.6851	9.9909	20
50	9.3119	9.3212	0.6788	9.9907	10
12° 0'	9.3179	9.3275	0.6725	9.9904	78° 0'
10	9.3238	9.3336	0.6664	9.9901	50
20	9.3296	9.3397	0.6603	9.9899	40
30	9.3353	9.3458	0.6542	9.9896	30
40	9.3410	9.3517	0.6483	9.9893	20
50	9.3466	9.3576	0.6424	9.9890	10
13° 0'	9.3521	9.3634	0.6366	9.9887	77° 0'
10	9.3575	9.3691	0.6309	9.9884	50
20	9.3629	9.3748	0.6252	9.9881	40
30	9.3682	9.3804	0.6196	9.9878	30
40	9.3734	9.3859	0.6141	9.9875	20
50	9.3786	9.3914	0.6086	9.9872	10
14° 0'	9.3837	9.3968	0.6032	9.9869	76° 0'
10	9.3887	9.4021	0.5979	9.9866	50
20	9.3937	9.4074	0.5926	9.9863	40
30	9.3986	9.4127	0.5873	9.9859	30
40	9.4035	9.4178	0.5822	9.9856	20
50	9.4083	9.4230	0.5770	9.9853	10
15° 0'	9.4130	9.4281	0.5719	9.9849	75° 0'
Angle	Log cos		Log tan	Log sin	Angle

Logarithms of Trigonometric Functions

Angle	Log sin	Log tan		Log cos	Angle
15° 0′	9 4190	9 4281	0 5719	9 9849	75° 0
10	9.4177	9 4331	0 5669	9 9846	50
20	9 4223	9 4381	0 5619	9 9943	40
30	9 4269	9 4430	0 5570	9 9839	80
40	9 4314	9 4479	0 5521	9 9836	20
50	9 4359	9 4527	0 6473	9 9932	90
10° 0	9 4403	9 4575	0 5425	9 9828	74° 0
10	9 4447	9 4622	0 5878	9 9825	50
20	9 4491	9 4669	0 5331	9 9821	40
30	9 4533	9 4716	0 5284	9 9817	30
40	9 4579	9 4762	0 5238	9 9814	20
60	9 4618	9 4809	0 5192	9 9910	10
17° 0	9 4859	9 4853	0 5147	9 9806	73° 0
10	9 4700	9 4898	0 5102	9 9802	50
20	9 4741	9 4943	0 5057	9 9798	40
80	9 4781	9 4987	0 5013	9 9794	50
40	9 4321	9 5031	0 4969	9 9790	20
90	9 4561	9 5075	0 4925	9 9786	10
18° 0′	9 4900	9 5119	0 4982	9 9782	72° 0
10	9 4939	9 8161	0 4839	9 9778	50
20	9 4977	9 5203	0 4797	9 9774	40
50	9 5015	9 5249	0 4765	9 9770	30
40	9 5052	9 9287	0 4713	9 9765	20
60	9 5090	9 5329	0 4671	9 9791	10
19° 0	9 5126	9 5370	0 4630	9 975	71° 0
10	9 5163	9 5411	0 4589	9 9762	50
20	9 5199	9 6451	0 4549	9 9748	90
30	9 8235	9 5491	0 4509	9 9743	50
40	9 5270	9 5531	0 4469	9 9739	20
60	9 5309	9 5571	0 4429	9.9734	10
20° 0	9 5341	9 5911	0 4389	9 9730	70° 0′
10	9 5375	9 5950	0 4350	9 9725	50
20	9 5409	9 5689	0 4311	9.9721	40
80	9 6443	9 9727	0 4273	9 9716	50
40	9 5477	9 5769	0 4234	9 9711	20
50	9 5510	9 5804	0 4196	9.9706	10
21° 0′	9 5543	9 5842	0 4158	9 9702	69° 0′
10	9 5575	9 5879	0.4121	9 9697	50
20	9 5689	9 5917	0 4083	9 9692	40
50	9 5641	9 9954	0 4046	9.9687	30
40	9 5873	9 5991	0 4009	9 9992	20
50	9 9704	9 6028	0.3972	9.9977	10
22° 0′	9 5739	9 6064	0 9936	9 9672	68° 0
10	9 5797	9 9190	0.3909	9 969	30
20	9 5798	9 9186	0.3854	9 9581	40
80	9 5829	9 9172	0.3929	9.9656	30
Angle	Log cos		Log tan	Log sin	Angle

Angle	Log sin	Log tan		Log cos	Angle
22° 30′	9 6828	9 9172	9 3828	9 9656	67° 30′
40	9 5859	9 6208	0.3792	9 9651	20
50	9 5889	9 6243	0.3757	9 9646	10
23° 0	9 5919	9 6279	9 8721	9 9640	67° 0
10	9 6948	9 6314	9.3686	9 9635	50
20	9 6978	9 6348	0 3652	9 9629	40
30	9 6007	9 6393	0.3617	9 9624	30
40	9 6036	9 6417	9 3583	9 9618	20
50	9 6055	9 6452	0 3549	9 9613	10
24° 0′	9 6093	9 6486	0 3514	9.9607	66° 0
10	9 9121	9 9520	0 5480	9 9592	50
20	9 6149	9 6553	0 8447	9 9596	40
50	9 6177	9 6597	0 3413	9 9590	30
40	9 6205	9 6620	0 3390	9 9584	20
50	9 6292	9 6654	0 3349	9 9579	10
25° 0′	9 6259	9 6687	0 3319	9 9573	05° 0
10	9 6286	9 6720	0 3280	9 9597	50
20	9 6313	9 6752	0 9249	9 9561	40
30	9 6340	9 8 95	0 3215	9 9655	50
40	9 6366	9 6817	0 3193	9 9649	20
50	9 6392	9 6850	0 3150	9 9543	10
26° 0	9 6418	9 6892	0 3118	9 953	64° 0
10	9 6444	9 6914	0.8096	9 9530	50
20	9 6470	9 6946	0.3054	9 9524	40
30	9 6495	9 6977	0 3023	9 9518	30
40	9 9521	9 7009	0 2991	9 9512	20
50	9 9546	9 7040	0.2960	9.9.05	10
27° 0′	9 6570	9 7072	0 2929	9 9499	63° 0
10	9 6595	9 7103	0 2997	9 9492	50
20	9 6629	9 7194	0.2965	9.9486	40
30	9 6544	9 7165	0 2835	9 9479	50
40	9 6668	9 7199	0.2804	9 9473	20
50	9 6692	9 7226	0.2774	9 9466	10
28° 0′	9 9716	9 7257	0.2749	9 9459	62° 0
10	9 9740	9 7297	0.2719	9 9453	59
20	9 9763	9 7817	0.2683	9.9446	40
30	9 9787	9 7348	0 2652	9 9439	30
40	9 5810	9 7978	0.2622	9 9432	29
50	9 6833	9 7408	9.2592	9 9425	10
29° 0′	9 5896	9 7438	0.2562	9 9418	01° 0
10	9 6879	9 7497	0.2533	9 9411	50
20	9 6901	9 7497	0.2503	9 9404	40
30	9 6923	9 7529	0.2474	9 9397	30
40	9 6946	9 7556	0.2444	9.9390	20
50	9 9968	9 7583	0.2419	9.9383	10
30° 0′	9 6990	9 7614	9.2386	9.9375	60° 0′
Angle	Log cos		Log tan	Log sin	Angle

Logarithms of Trigonometric Functions

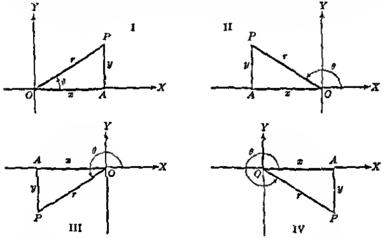
ANGLE	LOG SIN	LOG TAN		LOG COS	ANGLE
30° 0'	9.6990	9.7614	0.2386	9.9375	60° 0'
10	9.7012	9.7644	0.2356	9.9368	50
20	9.7033	9.7673	0.2327	9.9361	40
30	9.7055	9.7701	0.2299	9.9353	30
40	9.7076	9.7730	0.2270	9.9346	20
50	9.7097	9.7759	0.2241	9.9338	10
31° 0'	9.7118	9.7788	0.2212	9.9331	59° 0'
10	9.7139	9.7816	0.2184	9.9323	50
20	9.7160	9.7845	0.2155	9.9315	40
30	9.7181	9.7873	0.2127	9.9308	30
40	9.7201	9.7902	0.2098	9.9300	20
50	9.7222	9.7930	0.2070	9.9292	10
32° 0'	9.7242	9.7958	0.2042	9.9284	58° 0'
10	9.7262	9.7986	0.2014	9.9276	50
20	9.7282	9.8014	0.1986	9.9268	40
30	9.7302	9.8042	0.1958	9.9260	30
40	9.7322	9.8070	0.1930	9.9252	20
50	9.7342	9.8097	0.1903	9.9244	10
33° 0'	9.7361	9.8125	0.1875	9.9236	57° 0'
10	9.7380	9.8153	0.1847	9.9228	50
20	9.7400	9.8180	0.1820	9.9219	40
30	9.7419	9.8208	0.1792	9.9211	30
40	9.7438	9.8235	0.1765	9.9203	20
50	9.7457	9.8263	0.1737	9.9194	10
34° 0'	9.7476	9.8290	0.1710	9.9186	56° 0'
10	9.7494	9.8317	0.1683	9.9177	50
20	9.7513	9.8344	0.1656	9.9169	40
30	9.7531	9.8371	0.1629	9.9160	30
40	9.7550	9.8398	0.1602	9.9151	20
50	9.7568	9.8425	0.1575	9.9142	10
35° 0'	9.7586	9.8452	0.1548	9.9134	55° 0'
10	9.7604	9.8479	0.1521	9.9125	50
20	9.7622	9.8506	0.1494	9.9116	40
30	9.7640	9.8533	0.1467	9.9107	30
40	9.7657	9.8559	0.1441	9.9098	20
50	9.7675	9.8586	0.1414	9.9089	10
36° 0'	9.7692	9.8613	0.1387	9.9080	54° 0'
10	9.7710	9.8639	0.1361	9.9070	50
20	9.7727	9.8666	0.1334	9.9061	40
30	9.7744	9.8692	0.1308	9.9052	30
40	9.7761	9.8718	0.1282	9.9042	20
50	9.7778	9.8745	0.1255	9.9033	10
37° 0'	9.7795	9.8771	0.1229	9.9023	53° 0'
10	9.7811	9.8797	0.1203	9.9014	50
20	9.7828	9.8824	0.1176	9.9004	40
30	9.7844	9.8850	0.1150	9.8995	30
ANGLE	LOG COS		LOG TAN		ANGLE

ANGLE	LOG SIN	LOG TAN		LOG COS	ANGLE
37° 30'	9.7844	9.8850	0.1150	9.8995	52° 30'
40	9.7861	9.8876	0.1124	9.8985	20
50	9.7877	9.8902	0.1098	9.8975	10
38° 0'	9.7893	9.8928	0.1072	9.8965	52° 0'
10	9.7910	9.8954	0.1046	9.8955	50
20	9.7926	9.8980	0.1020	9.8945	40
30	9.7941	9.9006	0.0994	9.8935	30
40	9.7957	9.9032	0.0968	9.8925	20
50	9.7973	9.9058	0.0942	9.8915	10
39° 0'	9.7989	9.9084	0.0916	9.8905	51° 0'
10	9.8004	9.9110	0.0890	9.8895	50
20	9.8020	9.9135	0.0865	9.8884	40
30	9.8035	9.9161	0.0839	9.8874	30
40	9.8050	9.9187	0.0813	9.8864	20
50	9.8066	9.9212	0.0788	9.8853	10
40° 0'	9.8081	9.9238	0.0762	9.8843	50° 0'
10	9.8096	9.9264	0.0736	9.8832	50
20	9.8111	9.9289	0.0711	9.8821	40
30	9.8125	9.9315	0.0685	9.8810	30
40	9.8140	9.9341	0.0659	9.8800	20
50	9.8155	9.9366	0.0634	9.8789	10
41° 0'	9.8169	9.9392	0.0608	9.8778	49° 0'
10	9.8184	9.9417	0.0583	9.8767	50
20	9.8198	9.9443	0.0557	9.8756	40
30	9.8213	9.9468	0.0532	9.8745	30
40	9.8227	9.9494	0.0506	9.8733	20
50	9.8241	9.9519	0.0481	9.8722	10
42° 0'	9.8255	9.9544	0.0456	9.8711	48° 0'
10	9.8269	9.9570	0.0430	9.8699	50
20	9.8283	9.9595	0.0405	9.8688	40
30	9.8297	9.9621	0.0379	9.8676	30
40	9.8311	9.9646	0.0354	9.8665	20
50	9.8324	9.9671	0.0329	9.8653	10
43° 0'	9.8338	9.9697	0.0303	9.8641	47° 0'
10	9.8351	9.9722	0.0278	9.8629	50
20	9.8365	9.9747	0.0253	9.8618	40
30	9.8378	9.9772	0.0228	9.8606	30
40	9.8391	9.9798	0.0202	9.8594	20
50	9.8405	9.9823	0.0177	9.8582	10
44° 0'	9.8418	9.9848	0.0152	9.8569	46° 0'
10	9.8431	9.9874	0.0126	9.8557	50
20	9.8444	9.9899	0.0101	9.8545	40
30	9.8457	9.9924	0.0076	9.8532	30
40	9.8469	9.9949	0.0051	9.8520	20
50	9.8482	9.9975	0.0025	9.8507	10
45° 0'	9.8495	0.0000	0.0000	9.8495	45° 0'
ANGLE	LOG COS		LOG TAN	LOG SIN	ANGLE

An angle is said to be in the quadrant in which its terminal side lies Thus angles of 95° 460°, and $-200°$ are second quadrant angles, and angles of 200°, $-100°$ and 550° are third quadrant angles

Trigonometric Functions of Any Angle

Let θ (theta pronounced the ta) be any angle in standard position P any point on the terminal side OP of the $\angle AOP$ (except O), and $PA \perp$ to the x axis The drawings below show P in the four quadrants



We now define the six trigonometric functions in terms of rectangular co-ordinates as follows

$$\sin\theta = \frac{\text{ordinate}}{\text{radius vector}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{radius vector}}{\text{ordinate}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{abscissa}}{\text{radius vector}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{radius vector}}{\text{abscissa}} = \frac{r}{x}$$

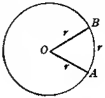
$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x} \qquad \text{ctn}\,\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}$$

You may not be familiar with the ratios of the second column above The cosecant (kō sē'kănt) of θ abbreviated csc θ is the reciprocal of sin θ, the secant (sē'kănt) of θ abbreviated sec θ is the reciprocal of cos θ and the cotangent (kō tăn'jĕnt) of θ abbreviated *ctn θ, is the reciprocal of tan θ*

Note that these definitions apply to angles of any size

The Radian

A *radian* is an angle which, when it is a central angle of a circle, intercepts an arc of the circle equal in length to the length of the radius In the figure at the right $\angle AOB$ is a radian Since $C = 2\pi r$, there are 2π radians in 360° Why? Then π radians = 180° and $\frac{\pi}{2}$ radians = 90°



EXERCISES

1. In Fig 1, find the value of $\sin \theta$, $\cos \theta$, $\tan \theta$, $\operatorname{ctn} \theta$, $\sec \theta$

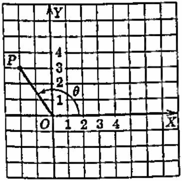


Fig 1

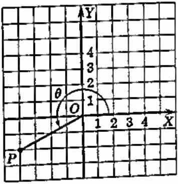


Fig 2

2. In Fig 2, find the value of $\sin \theta$, $\cos \theta$, $\tan \theta$, $\csc \theta$

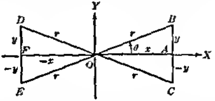
3. Express the following angles in radians, using π in each answer

a. 90°	d. 360°	g. 135°
b. 180°	e. 60°	h 720°
c. 270°	f. 45°	i 30°

Reference Angle

Let $\angle AOB = \theta$ be any first quadrant angle in standard position $\angle AOD$, AOE, and AOC are positive angles equal to $180° - \theta$, $180° + \theta$, and $360° - \theta$ respectively The angles $180° - \theta$, $180° + \theta$, and

$360° - \theta$ are said to be referred to the x-axis and θ is the *reference angle*. Take OB, OD, OE, and OC each equal to r. Then right $\triangle AOB$, DOF, EOF, and AOC are congruent. The abscissas of points B, D, E and C have the same absolute values (see page 8 in the text), and the ordinates of these points have the same absolute values. A study of the figure thus formed shows that



To find a function of any given angle between 90° and 360° (except 180° or 270°) express the angle as $(180° - \theta)$, $(180° + \theta)$, or $(360° - \theta)$, then take the same function of the reference angle and give it the sign determined by the function of the given angle in the quadrant to which it belongs.

Example Find the cos of 190°.

Solution
$\cos 190° = \cos (180° + 10°)$
$\cos 190° = -\cos 10°$
$\cos 190° = -.9848$

EXERCISES

1 Express the following as functions of positive acute angles

a $\sin 170°$ e $\tan 195°$ i $\sin 330° 10'$
b $\csc 135°$ f $\cos 212°$ j $\cos 291° 6'$
c $\operatorname{ctn} 170°$ g $\csc 230°$ k $\tan 350° 40'$
d $\cos 135°$ h $\sec 185°$ l $\csc 295° 56'$

2 Find the numerical value of each of the following

a $\sin 110°$ c $\tan 332°$ e $\sin 311° 20'$
b $\cos 225°$ d $\tan 256° 40'$ f $\cos 333° 11'$

Inverse Functions

In the preceding section we learned how to find the function of a particular angle. In this section we shall learn how to find angles having a particular function. Thus, to find the positive angles less

than 360° whose tangents are -1, we observe that the tangent of an angle is negative in the second and fourth quadrants Since the reference angle associated with the tangent 1 is 45° we know that the required angles are 135° and 315°

We often abbreviate the expression "y is the angle whose tangent is x," by writing $y = \text{arc tan } x$, or sometimes $y = \tan^{-1} x$ (-1, in this case, is not an exponent) Similarly, to express the thought "y is the angle whose sine is x," we write $y = \text{arc sin } x$ or $y = \sin^{-1} x$ What is the meaning of each of the following $y = \text{arc cos } x$, $y = \text{arc ctn } x$, $y = \text{arc sec } x$, $y = \text{arc csc } x$? We sometimes call these functions the *inverse trigonometric functions*

EXERCISES

1 Find the positive angles less than 360° whose

a sines are $\frac{1}{2}$
b cosines are $-$ 9397
c tangents are 9 7882
d sines are $-$ 6439

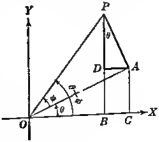
2 Find the positive angles less than 360° represented by

a arc tan 1
b arc sin $-\frac{1}{2}$
c $\tan^{-1} -\frac{4}{3}$
d arc cos $-\frac{1}{2}$

The Sine and the Cosine of the Sum of Two Angles

Let us now consider the sine and the cosine of the sum of two angles Although the formulas we shall derive hold for any two angles, we shall consider only angles whose sum is less than a right angle

In the figure, let θ, ϕ (fi), and $(\theta + \phi)$ be acute angles θ is in standard position, and the initial side of ϕ coincides with the terminal side of θ From P, any point on the terminal side of $(\theta + \phi)$, draw PA perpendicular to the initial side of ϕ and PB perpendicular to the initial side of θ From A draw AD perpendicular to PB, and AC perpendicular to OB Then $\angle APD = \theta$ Why? $DA = BC$ Why? By definition,



$$\sin(\theta + \phi) = \frac{PB}{OP} = \frac{AC + PD}{OP} = \frac{AC}{OP} + \frac{PD}{OP}$$

The ratios $\frac{AC}{OP}$ and $\frac{PD}{OP}$ are not functions of either θ or ϕ To obtain

functions of these angles, we multiply $\frac{AC}{OP}$ by $\frac{OA}{OA}$, and we multiply $\frac{PD}{OP}$ by $\frac{PA}{PA}$ (Note that in each case we are multiplying by 1) Then,

$$\sin(\theta+\phi)=\frac{AC}{OP}\cdot\frac{OA}{OA}+\frac{PD}{OP}\cdot\frac{PA}{PA}=\frac{AC}{OA}\cdot\frac{OA}{OP}+\frac{PD}{PA}\cdot\frac{PA}{OP}$$

Therefore, $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$

From the figure on page 577 we also have, by definition,

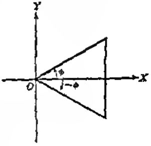
$$\cos(\theta+\phi)=\frac{OB}{OP}=\frac{OC-DA}{OP}=\frac{OC}{OP}-\frac{DA}{OP}$$
$$=\frac{OC}{OP}\cdot\frac{OA}{OA}-\frac{DA}{OP}\cdot\frac{PA}{PA}=\frac{OC}{OA}\cdot\frac{OA}{OP}-\frac{DA}{PA}\cdot\frac{PA}{OP}$$

Therefore, $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$

The Sine and the Cosine of the Difference of Two Angles

Since the addition formulas are true for all angles, positive or negative, the subtraction formulas may be derived as follows

$$\sin(\theta-\phi)=\sin[\theta+(-\phi)]$$
$$=\sin\theta\cos(-\phi)+\cos\theta\sin(-\phi)$$



A glance at the sketch at the right shows us that $\cos\phi=\cos(-\phi)$, and $\sin(-\phi)=-\sin\phi$ Substituting these values in the equation above, we have

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

In like manner,

$$\cos(\theta-\phi)=\cos[(\theta+(-\phi)]$$
$$=\cos\theta\cos(-\phi)-\sin\theta\sin(-\phi)$$

or $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$

EXERCISES

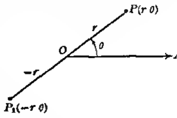
1 Using the functions of 30° and 45° and the addition formula, find the sine and cosine of 75°

2 Using the functions of 45° and 60° and the subtraction formula find the sine and cosine of 15°

3 Simplify **a** $\sin(x+30°)$ **b** $\cos(60°-A)$

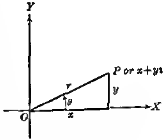
Polar Co-ordinates

In the polar system of co-ordinates the position of a point is described with reference to a fixed point, O, called the *pole* or *origin*, and a fixed, positively directed line, OA, called the *polar axis*, or *initial line*. The position of point P is given by (r, θ), called *polar co-ordinates*. The angle θ, when measured counterclockwise from OA to OP is positive and when measured clockwise from OA to OP is negative. When r lies on the terminal side of θ it is positive, but when r lies on the extension of the terminal side through the pole it is negative. Thus the co-ordinates of P_1 may be given as $(-r, \theta)$. Since any point in the plane can be designated by positive polar co-ordinates, we shall restrict our work to positive values of r and θ.



Polar Form of a Complex Number

Since we frequently need to change from one system of co-ordinates to the other, it is customary to superimpose the rectangular system on the polar system. In the figure, O is the origin of both systems, OX is the polar axis of the polar system, and the y-axis is referred to as the $\frac{\pi}{2}$ axis of the polar system. Do you see why the y axis is called the $\frac{\pi}{2}$ axis? The complex number $x + yi$ is represented by $P(x, y)$ in the rectangular system and by $P(r, \theta)$ in the polar system.



Then $$\frac{x}{r} = \cos\theta \quad \text{or} \quad x = r\cos\theta \tag{1}$$

and $$\frac{y}{r} = \sin\theta \quad \text{or} \quad y = r\sin\theta \tag{2}$$

If we multiply both members of equation (2) by i we have

$$yi = ri\sin\theta \tag{3}$$

Adding (1) and (3), $x + yi = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$. *The expression $r(\cos\theta + i\sin\theta)$ is the polar form of the complex number $x + yi$.* r is the *modulus* or *absolute value* of the complex number and θ is the *amplitude*. r and θ are taken as positive. $\cos\theta + i\sin\theta$ may be abbreviated by the symbol cis θ. Why cis?

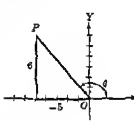
Thus $$r(\cos\theta + i\sin\theta) = r \text{ cis } \theta$$

To express a complex number in polar form, we have, from the figure

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \text{arc tan } \frac{y}{x}$$

Example 1 Express $-5 + 6i$ in polar co-ordinates.

Solution $-5 + 6i$ is represented by the point P in the figure.



$x = -5$ and $y = 6$

$r = \sqrt{x^2 + y^2} = \sqrt{25 + 36}$

$= \sqrt{61}$

$\theta = \text{arc tan } \frac{6}{-5}$

$= \text{arc tan } -1.2000$

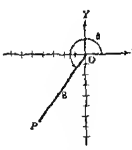
The reference angle $= \text{arc tan } 1.2000$ or $50° 12'$

$\theta = 180° - 50° 12' = 129° 48'$

Then $r \text{ cis } \theta = \sqrt{61} \text{ cis } 129° 48'$

Example 2 Express 8 cis 240° in rectangular form.

Solution Represent 8 cis 240° by point P.



$x = r\cos(180° + 60°)$

$= 8(-.5) = -4$

$y = r\sin\theta = 8\sin 240°$

$= 8\sin(180° + 60°)$

$= 8(-\frac{1}{2}\sqrt{3}) = -4\sqrt{3}$

Then $x + iy = -4 - 4i\sqrt{3}$

EXERCISES

Plot the points which represent each of the following and express in polar form.

1. $1 - i$	3. $-1 + 3i$	5. $-7 - 2i$	7. -3
2. $4 + 3i$	4. $\sqrt{5} - i$	6. $i + 4\sqrt{3}$	8. $2i$

Plot the points which represent each of the following and express in rectangular form

9. $5 \text{ cis } 45°$ 11. $6 \text{ cis } 270°$ 13. $\text{cis } 135°$ 15. $4 \text{ cis } 330°$

10. $3 \text{ cis } 210°$ 12. $2 \text{ cis } 120°$ 14. $2 \text{ cis } 30°$ 16. $3 \text{ cis } 0°$

The Product of Two Complex Numbers in Polar Form

Let $\quad z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

and $\quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

be any two complex numbers in polar form

Then, by multiplication, we have

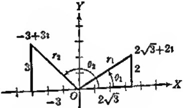
$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &\quad + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] = r_1 r_2 \text{ cis } (\theta_1 + \theta_2) \end{aligned}$$

This may be stated in the form of a theorem as follows

The product of two complex numbers is a complex number whose modulus is the product of the moduli and whose amplitude is the sum of the amplitudes of the given complex numbers.

Example. Find the product of $2\sqrt{3} + 2i$ and $-3 + 3i$

Solution. Changing $2\sqrt{3} + 2i$ to polar form we have



$$r_1 = \sqrt{12 + 4} = 4 \text{ and}$$

$$\theta_1 = \text{arc tan } \frac{1}{\sqrt{3}} = 30°$$

Then $2\sqrt{3} + 2i = 4 \text{ cis } 30°$

Changing $-3 + 3i$ to polar form,

we have $r_2 = \sqrt{9+9} = 3\sqrt{2}$ and $\theta_2 = \text{arc tan } (-1) = 135°$

Then $-3 + 3i = 3\sqrt{2} \text{ cis } 135°$

Hence $(2\sqrt{3} + 2i)(-3 + 3i) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

$= 12\sqrt{2} \text{ cis } 165°$

The Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

be any two complex numbers in polar form

Then, by division, we have $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$.

Multiplying both numerator and denominator of the right member by $\cos \theta_2 - i \sin \theta_2$ we have

$$\frac{z_1}{z_2} = \frac{r_1[(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)]}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)}$$

$$= \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)] = \frac{r_1}{r_2} \text{cis} (\theta_1 - \theta_2)$$

This may be stated in the form of a theorem as follows

The quotient of two complex numbers is a complex number whose modulus is the modulus of the dividend divided by the modulus of the divisor and whose amplitude is the amplitude of the dividend minus the amplitude of the divisor

Example Divide $6 \text{ cis } 150°$ by $2\sqrt{3} \text{ cis } 60°$

Solution $\dfrac{6 \text{ cis } 150°}{2\sqrt{3} \text{ cis } 60°} = \dfrac{6}{2\sqrt{3}} \text{ cis } 90° = \sqrt{3} \text{ cis } 90°$

DeMoivre's Theorem

If we use the theorem for the product of two complex numbers to multiply $r(\cos \theta + i \sin \theta)$ successively by itself, we have

$$[r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta)$$
$$[r(\cos \theta + i \sin \theta)]^3 = r^3(\cos 3\theta + i \sin 3\theta)$$

and so on to $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) = r^n \text{ cis } n\theta$

This may be stated in the form of a theorem as follows

The nth power of a complex number is a complex number whose modulus is the nth power of the modulus of the given number and whose amplitude is n times the amplitude of the given number

This theorem is known as *DeMoivre's Theorem* The theorem is also valid when *n is negative or fractional*

Example Find $(\sqrt{2} + i\sqrt{2})^4$

Solution Writing $\sqrt{2} + i\sqrt{2}$ in polar form, we have $2 \text{ cis } 45°$
Then $[2 \text{ cis } 45°]^4 = 2^4 \text{ cis } 180° = 16(-1 + 0) = -16$

EXERCISES

Perform the indicated operations using the polar form of the complex numbers

1 $2 \text{ cis } 30° \cdot 4 \text{ cis } 120°$

2 $5 \text{ cis } 25° \cdot 3 \text{ cis } 40°$

3 $3 \text{ cis } 120° \div \text{cis } 30°$

4 $9 \text{ cis } 210° \div 3 \text{ cis } 30°$

5 $(1-i)(\sqrt{2}+i\sqrt{2})$ 7 $(4-i\sqrt{3})^3$
6 $(1+i)^2$ 8 $(2-i)\div(3+i)$

Roots of Complex Numbers

Since DeMoivre's Theorem is valid when n is fractional we shall replace n by $\frac{1}{n}$ in the formula Then

$$[r(\cos\theta+i\sin\theta)]^{\frac{1}{n}}=r^{\frac{1}{n}}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)$$

If m is a positive integer

$$\cos(\theta+m\,360^\circ)=\cos\theta \text{ and } \sin(\theta+m\,360^\circ)=\sin\theta$$

Then

$$[r(\cos\theta+i\sin\theta)]^{\frac{1}{n}}=r^{\frac{1}{n}}\left(\cos\frac{\theta+m\,360^\circ}{n}+i\sin\frac{\theta+m\,360^\circ}{n}\right)$$

This formula can be used to find all the nth roots of any number

Example 1 Find the cube root of $-4+4i\sqrt{3}$

Solution $-4+4i\sqrt{3}=8(\cos 120^\circ+i\sin 120^\circ)$
Then $[r(\cos\theta+i\sin\theta)]^{\frac{1}{3}}=[8(\cos 120^\circ+i\sin 120^\circ)]^{\frac{1}{3}}$
$=2(\cos 40^\circ+i\sin 40^\circ)$

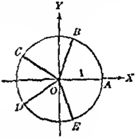
This is one cube root of $-4+4i\sqrt{3}$ But since the trigonometric functions of 120° are the same as those for 120° plus any integral multiple of 360° we may write the cube roots of $-4+4i\sqrt{3}$ as $(8\text{ cis }120^\circ)^{\frac{1}{3}}$ $(8\text{ cis }480^\circ)^{\frac{1}{3}}$ $(8\text{ cis }840^\circ)^{\frac{1}{3}}$, and so on However, when we evaluate these we find that after the first three the values are repeated in cycles of three This proves that the number $-4+4i\sqrt{3}$ has three and only three distinct cube roots The other two cube roots of $-4+4i\sqrt{3}$ in polar form are $(8\text{ cis }480^\circ)^{\frac{1}{3}}=2\text{ cis }160^\circ$ and $(8\text{ cis }840^\circ)^{\frac{1}{3}}=2\text{ cis }280^\circ$ We accept, without proof, the theorem

Every complex number except zero has n and only n distinct nth roots.

The n distinct nth roots of a number may be represented graphically as points on a circle whose center is at the origin and whose radius is the modulus of these roots These points divide the circle into n equal arcs and are the vertices of a regular polygon of n sides inscribed in the circle

Example 2 Find the five distinct fifth roots of 1 and represent the points graphically

Solution $1 = \text{cis } 0°, \text{cis } 360°, \text{cis } 720°, \text{cis } 1080°, \text{cis } 1440°$ Then the fifth roots of 1 are $\text{cis } 0°$, $\text{cis } 72°$, $\text{cis } 144°$, $\text{cis } 216°$, and $\text{cis } 288°$ The points A, B, C, D, and E represent the five fifth roots of 1 The moduli of the points are 1 $\widehat{AB} = \widehat{BC} = \widehat{CD} = \widehat{DE} = \widehat{E1} = 72°$ Proceeding to find the roots in rectangular form, we have



$$\text{cis } 0° = 1(\cos 0° + i \sin 0°) = 1$$
$$\text{cis } 72° = 1(\cos 72° + i \sin 72°)$$
$$= 3090 + 9511\,i$$
$$\text{cis } 144° = 1(\cos 144° + i \sin 144°)$$
$$= -8090 + 5878\,i$$
$$\text{cis } 216° = 1(\cos 216° + i \sin 216°)$$
$$= -8090 - 5878\,i$$
$$\text{cis } 288° = 1(\cos 288° + i \sin 288°) = 3090 - 9511\,i$$

Finding the 5 fifth roots is the same as solving the equation $x^5 - 1 = 0$ The solutions of the equation $x^5 - 1 = 0$ are the vertices of a regular polygon inscribed in the circle whose radius is 1

EXERCISES

Express by rectangular co-ordinates

1 $r = 2 \cos \theta$ 2 $r = \sin \theta$

Using the polar form of complex numbers find and plot

3 The three cube roots of 27

4 The three cube roots of $-i$

5 The four fourth roots of $1 + i$

Solve the following equations

6 $x^4 = 32$ 7 $x^3 - 1 = 0$ 8 $x^6 = -1$ 9 $x^4 - 8 = 0$

10. $x^6 - 1 = 0$ 11 $x^3 - 8 = 0$ 12 $x^3 + i = 0$ 13 $x^3 - i = 0$

14. Solve by two methods $x^3 - 27 = 0$

Using the polar form of complex numbers, perform the following indicated operations

15 $(\sqrt{3} - i)^6$ 16 $(2 - i\sqrt{3})^3$ 17. $(-1 + i\sqrt{2})^3$

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